Honors Problem Set Newton's Method

Draw pictures which illustrate each of the following behaviors of Newton's method.

P1. Let $f(c) = 0$. Suppose that $f(x)$ is positive, CCU, and increasing for $x > c$. Then Newton's method will converge from the right to the root c.

P2. Let $f(c) = 0$. Suppose that $f(x)$ is positive, CCU, and decreasing for $x < c$. Then Newton's method will converge from the left to the root c.

P3. Let $f(c) = 0$. Suppose that $f(x)$ is negative, CCD, and increasing for $x < c$. Then Newton's method will converge from the left to the root c.

P4. Let $f(c) = 0$. Suppose that $f(x)$ is negative, CCD, and decreasing for $x > c$. Then Newton's method will converge from the right to the root c.

We define the Newton function $N(x)$ to be the function that we iterate when using Newton's method to approximate roots of $f(x)$:

$$N(x) = x - \frac{f(x)}{f'(x)}.$$

You are to investigate what happens when we apply Newton's function repeatedly to a number x in the case where the underlying function is

$$f(x) = x^4 - x^2.$$

The following exercises will help guide this investigation.

E1. Sketch the graph of $f(x)$ indicating places where there are intercepts, local maxima, local minima and points of inflection.

E2. Show that the corresponding Newton function is the rational function

$$N(x) = \frac{x(3x^2 - 1)}{2(2x^2 - 1)}$$

In the next few exercises, you will show that there are five things that can happen to a number x under successive iterations $N^k(x)$ of the Newton function.

(a) Successive N-iterates of x converge to the root -1.

(b) Successive N-iterates of x converge to the root 0.

(c) Successive N-iterates of x converge to the root $+1$.

(d) The k-th iterate $N^k(x)$ becomes undefined for some k.

(e) The N-iterates of x oscillate between two values for ever.

E3. Show that if $x = \pm\sqrt{1/2}$, then $N(x)$ is undefined. [case (d) above]

E4. Show that all points in the interval $(\sqrt{1/2}, \infty)$ converge to the root 1. [case (c) above]

E5. Show that all points in the interval $(-\infty, -\sqrt{1/2})$ converge to the root -1. [case (a) above]

E6. Find the number a so that $N(a) = -a$. Verify that successive N-iterates of a or $-a$ simply oscillate back and forth between these two values. [case (e) above]

E7. Show that all points in $(-a, a)$ converge to the root 0. [case (b) above]

E8. **Challenge:** We've seen examples of all 5 types of behavior. We have also seen where almost all points on the real line end up after successive N-iterations. There are still two open intervals (open means endpoints are not included) that we have to understand:

$$(-\sqrt{1/2}, -a) \qquad \text{and} \qquad (a, \sqrt{1/2})$$

Claim: Each of these two intervals is broken up into infinitely many smaller open sub-intervals. Points in these sub-intervals are (alternately) in case (a) or case (c). The "endpoints" between adjacent open intervals all are in case (d).

You can ask for a hint if you do not see this. But you should first try to see why the claim might be true.

Remark. This exercise had you examine what mathematicians term a *dynamical system*. In this case the underlying set is the real line $\mathbb{R}$ and the underlying function is the Newton function $N(x)$ (which is just a rational function). The dynamical system consists of $\mathbb{R}$ together with all iterates (composites) N^k of N.

If we were to extend the universe of numbers beyond $\mathbb{R}$ to include objects like $\sqrt{-1}$, we would obtain the *complex numbers* $\mathbb{C}$. Working through problems 1–8 above with complex numbers yields beautiful *fractal* pictures. Google the phrase "Newton basin fractal" to see some pictures.

Dynamical systems consisting of iterations of rational maps over the complex numbers (iterated rational maps) have been the subject of intense mathematical activity since the 1920's. This area has attracted more than the average number of Fields medalists (math analogues of Nobel prize winners) over the years.