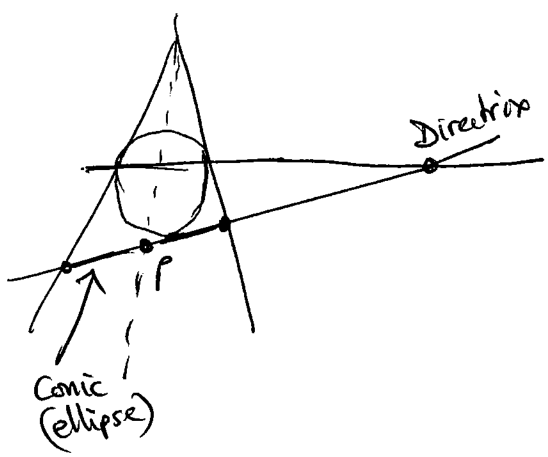
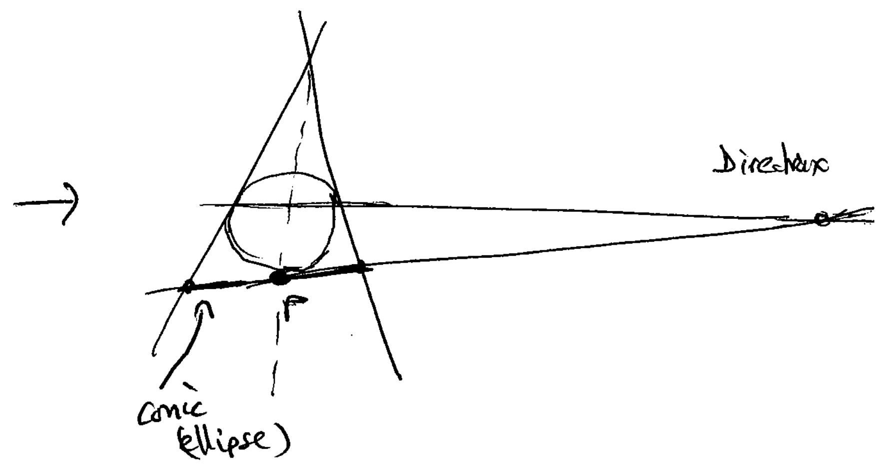
Calc III (Honors) Extra Work I.

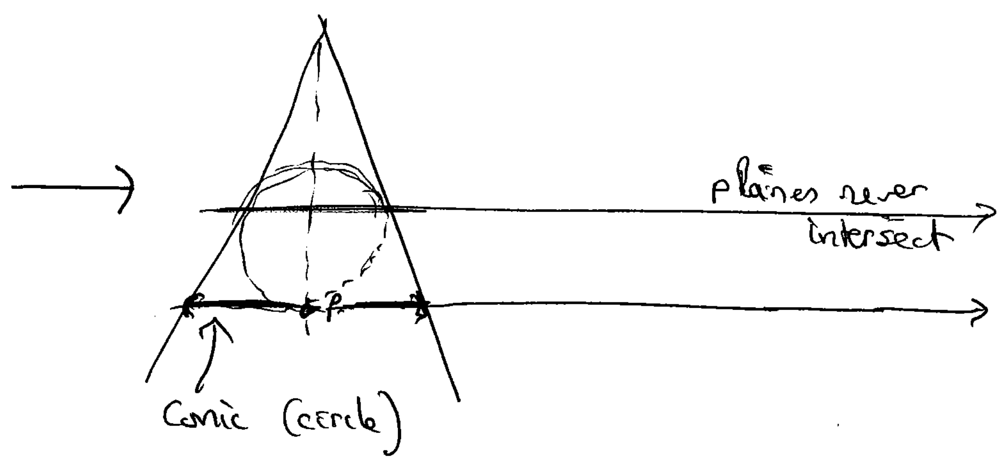
Q1, Q2 } check out web pages linked from course home page.
Q6, Q7 }

Q3 — see attached pages

<u>Q4</u> Fix a point P on cone axis. Consider slant planes through P. Slowly tilt planes until they limit on horizontal → (directrix line → ∞).







Q5 (Ellipse) Conic with eccentricity ϵ in polar coords

$$r = \frac{\varepsilon d}{1+\varepsilon\cos\theta}$$

$$r + \varepsilon r\cos\theta = \varepsilon d$$

$$\sqrt{x^2+y^2} + \varepsilon x = \varepsilon d$$

$$\sqrt{x^2+y^2} = \varepsilon d - \varepsilon x$$

$$x^2+y^2 = \varepsilon^2 d^2 + \varepsilon^2 x^2 - 2\varepsilon^2 d x$$

$$(1-\varepsilon^2)\,x^2 + 2\varepsilon^2 d\,x + y^2 = \varepsilon^2 d^2$$

$$(1-\varepsilon^2)\left[x^2 + 2\left(\frac{\varepsilon^2 d}{1-\varepsilon^2}\right)x + \left(\frac{\varepsilon^2 d}{1-\varepsilon^2}\right)^2\right] + y^2 = \varepsilon^2 d^2 + \frac{\varepsilon^4 d^2}{1-\varepsilon^2}$$

$$= \frac{\varepsilon^2 d^2}{1-\varepsilon^2}$$

$$\frac{\left(x + \frac{\varepsilon^2 d}{1-\varepsilon^2}\right)^2}{\left(\frac{\varepsilon d}{1-\varepsilon^2}\right)^2} + \frac{y^2}{\left(\frac{\varepsilon d}{\sqrt{1-\varepsilon^2}}\right)^2} = 1$$

Compare $\dfrac{(x+h)^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$a^2 \leftrightarrow \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2} \qquad b^2 \leftrightarrow \frac{\varepsilon^2 d^2}{1-\varepsilon^2}$$

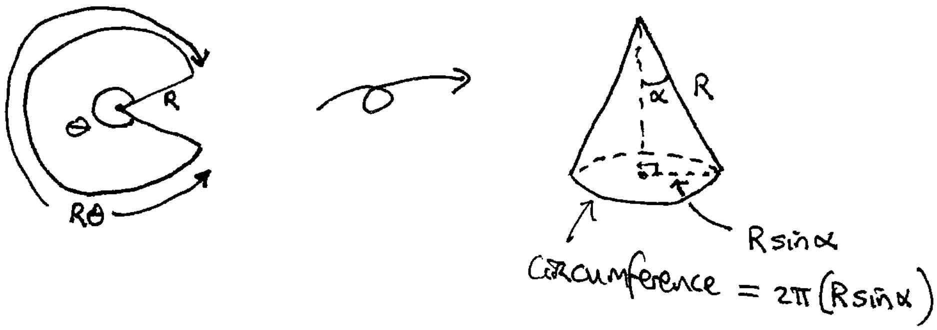
$$\frac{b^2}{a^2} = 1-\varepsilon^2 \Rightarrow \varepsilon^2 = 1 - \frac{b^2}{a^2} = \frac{a^2-b^2}{a^2} \Rightarrow$$

$*$

$$\boxed{\varepsilon = \frac{\sqrt{a^2-b^2}}{a} = \frac{c}{a}}$$

Constructing the Model... (details)

A Pacman Piece ⇝ Cone



$$\cancel{R}\theta = 2\pi \cancel{R}\sin\alpha \qquad \boxed{\sin(\alpha) = \frac{\theta}{2\pi}}$$

We'll work with pacman angle of $\pi \Rightarrow \sin(\alpha) = \frac{\pi}{2\pi} = \frac{1}{2} \Rightarrow \boxed{\alpha = \pi/6}$

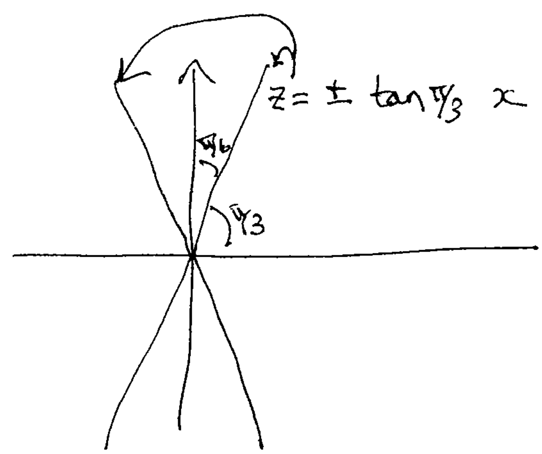
B Cone equation in 3-d.

$$z = \pm \tan(\pi/3)\sqrt{x^2+y^2}$$

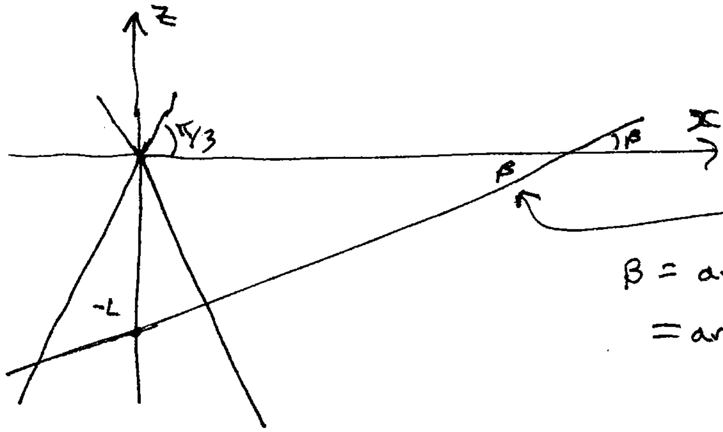
$$z = \pm \frac{\sqrt{3}}{1}\sqrt{x^2+y^2}$$

$$\boxed{z^2 = 3(x^2+y^2)}$$

Cone. —— (i)



C Ellipse plane in 3-d.



β = angle between line & x-axis
= angle between ellipse plane and xy-plane

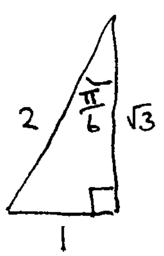
Ellipse plane $\boxed{z = (\tan(\beta))\,x - L}$ ← eqn on x, y, z.

slope intercept.

$$0 < \beta < \pi/3$$

We'll take $\beta = \pi/6$ and $L = 1$ for a specific example.

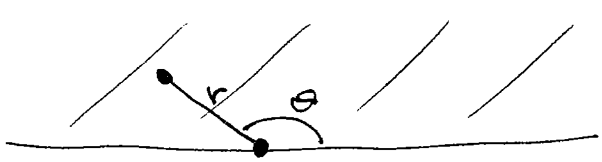
Can always "rescale" model by using whatever units we wish.

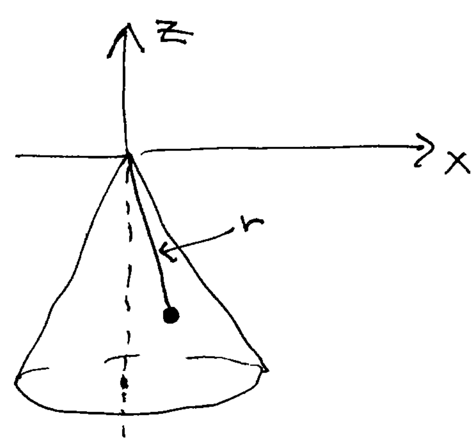


$\boxed{z = \dfrac{x}{\sqrt{3}} - 1}$ —— (ii) Ellipse plane

D Curve on pacman piece ⇝ ellipse on ellipse plane

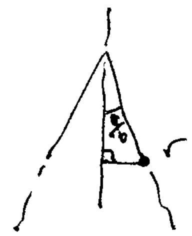
Use polar coords on pacman piece





segment of length r projects to z-axis to have length $r\cos(\pi/6)$

$$= \frac{\sqrt{3}\,r}{2}$$



It projects to xy-plane to have length $r\sin(\pi/6)$

$$= \frac{r}{2}$$

One sweep of Angle $0 \le \theta \le \pi$ on pacman plane $\rightsquigarrow$ 2π rotation (about z-axis) on cone.

So $\theta \longrightarrow 2\theta$ in map to cone.

$$(r,\theta) \longrightarrow \left(r\sin(\pi/6)\cos(2\theta),\ r\sin(\pi/6)\sin(2\theta),\ -r\cos(\pi/6)\right)$$

on pacman piece

$$= \left(\frac{r}{2}\cos(2\theta),\ \frac{r}{2}\sin(2\theta),\ -\frac{\sqrt{3}\,r}{2} \right)$$

check: this point satisfies eq^n (i).

Points on ellipse also satisfy equation (ii)

$$z = \frac{x}{\sqrt{3}} - 1$$

$$-\frac{\sqrt{3}\,r}{2} = \frac{r}{2\sqrt{3}}\cos(2\theta) - 1$$

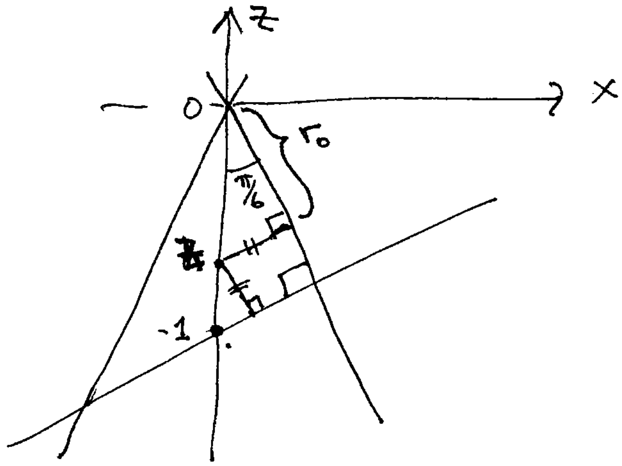
$$-3r = r\cos(2\theta) - 2\sqrt{3}$$

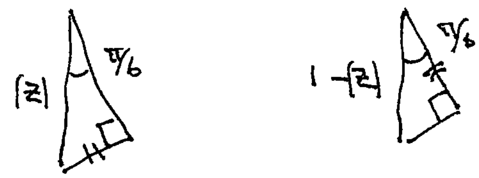
$$2\sqrt{3} = r\left(3 + \cos(2\theta)\right)$$

Curve on pacman piece which becomes ellipse of cone.

$$\boxed{r = \frac{2\sqrt{3}}{3 + \cos(2\theta)}} \quad \text{——— (iii)}$$

E Upper Dandelin circle (intersection of sphere with cone).



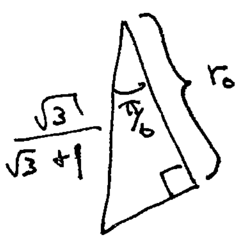


$$|z| \sin(\pi/6) = (1-|z|)\cos(\pi/6)$$

$$\frac{|z|}{\cancel{2}} = (1-|z|)\frac{\sqrt{3}}{\cancel{2}}$$

$$|z|(\sqrt{3}+1) = \sqrt{3}$$

$$|z| = \frac{\sqrt{3}}{\sqrt{3}+1}$$



$$\frac{r_0}{\frac{\sqrt{3}}{\sqrt{3}+1}} = \cos(\pi/6) = \frac{\sqrt{3}}{2} \Rightarrow r_0 = \frac{3}{2(\sqrt{3}+1)}$$

$$\boxed{r = \frac{3}{2(\sqrt{3}+1)}} \quad \text{—(iv)}$$

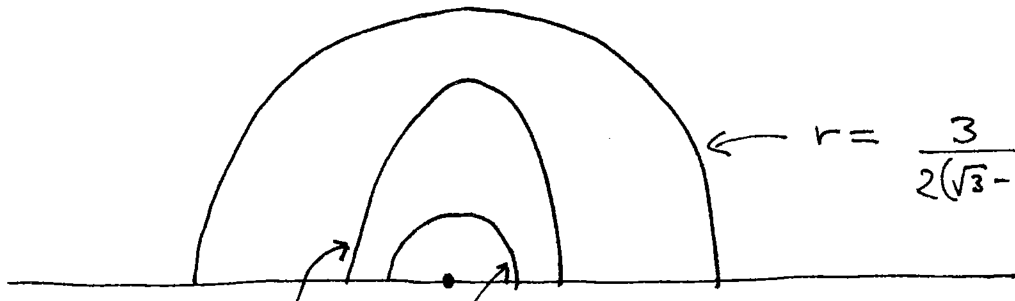
Polar coords circle on pacman piece ⇝ Dandelin circle on cone.

Remark: You should also prove that the circle (on pacman piece)

$$\boxed{r = \frac{3}{2(\sqrt{3}-1)}} \quad \text{—(v)}$$

becomes the lower Dandelin circle on the cone.

Summary (Pacman Piece is now finished!)



$r = \dfrac{3}{2(\sqrt{3}-1)}$ --- eqn (v)

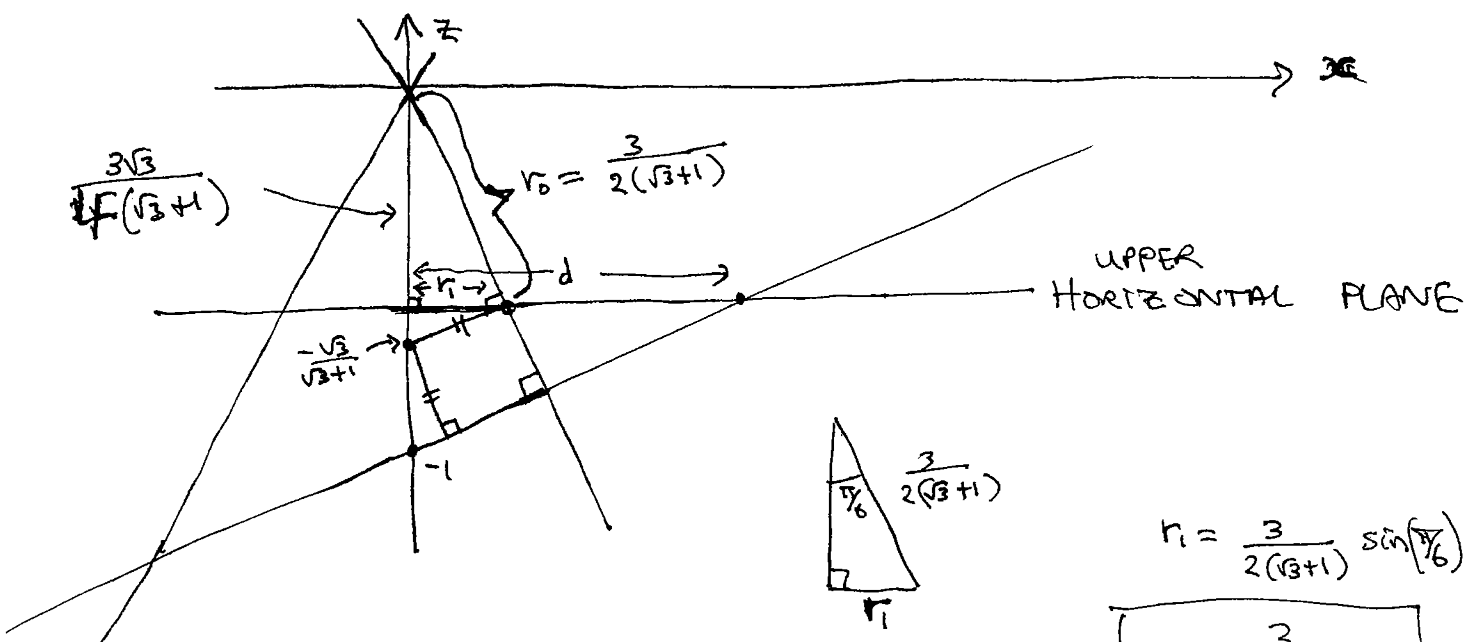
$r = \dfrac{2\sqrt{3}}{3+\cos(2\theta)}$ --- eqn (iii)

$r = \dfrac{3}{2(\sqrt{3}+1)}$ ---- eqn (iv)

[End of Remark]

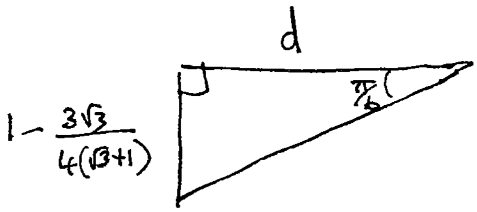
F Upper horizontal plane (containing upper Dandelin circle).

From section E we have ---



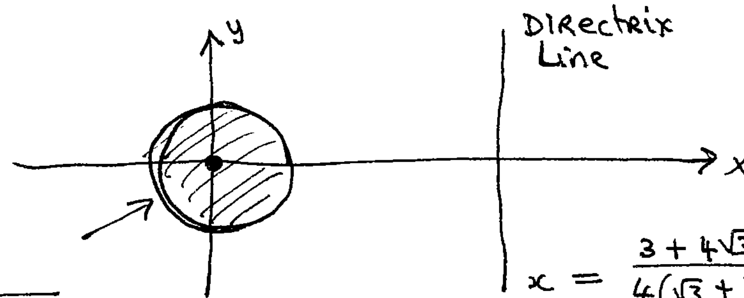
$r_1 = \dfrac{3}{2(\sqrt{3}+1)}\sin(\pi/6)$

$$\boxed{r_1 = \frac{3}{4(\sqrt{3}+1)}} \text{ — (vi)}$$



$$d = \sqrt{3}\left(1 - \frac{3\sqrt{3}}{4(\sqrt{3}+1)}\right) \Rightarrow \boxed{d = \frac{3+4\sqrt{3}}{4(\sqrt{3}+1)}} \text{ — (vii)}$$

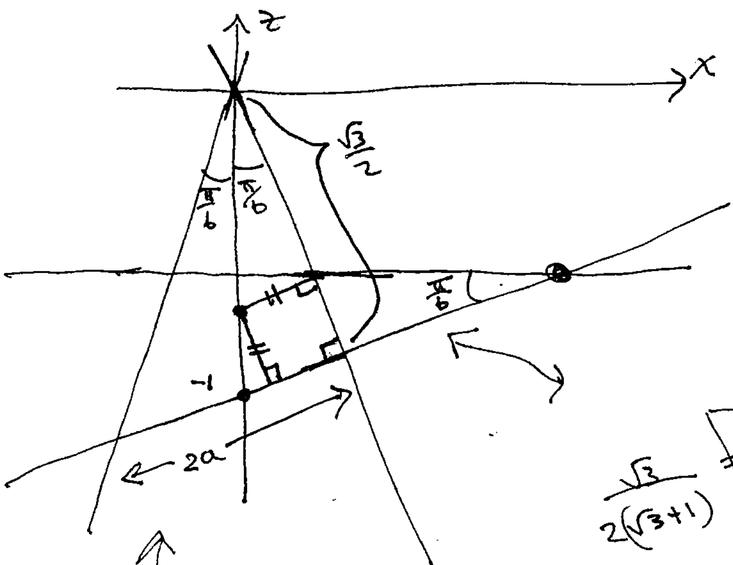
Summary (Upper Horizontal Plane finished!)



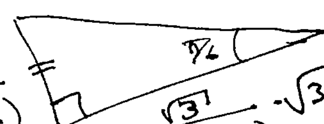
Center $=(0,0)$
Radius $= \frac{3}{4(\sqrt{3}+1)}$ by eq^n (vi)

$x = \frac{3+4\sqrt{3}}{4(\sqrt{3}+1)}$ --- by eq^n (vii)

G Ellipse Plane...

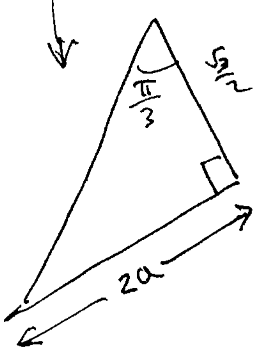


From section E we know length of ‖ segment $= \frac{|z|}{2} = \frac{\sqrt{3}}{2(\sqrt{3}+1)}$



$\frac{\sqrt{3}}{2(\sqrt{3}+1)}$ $\quad \frac{\sqrt{3}}{2(\sqrt{3}+1)}\cdot\sqrt{3} = \frac{3}{2(\sqrt{3}+1)}$

Add this to "a" to get directrix distance from $(0,0)$

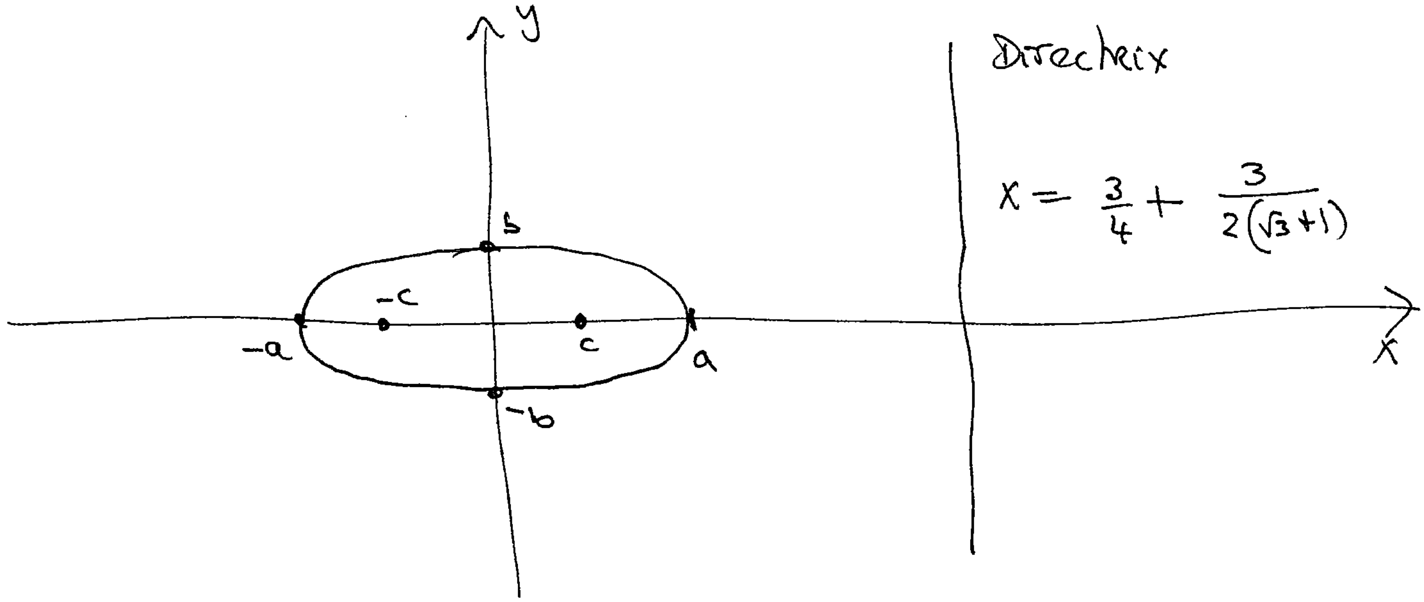


$$\Rightarrow 2a = \sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{3}{2} \Rightarrow \boxed{a = \frac{3}{4}}$$

semi major axis of ellipse

$$a + \frac{3}{2(\sqrt{3}+1)} = \text{distance from directrix to center of ellipse.}$$

Summary (ellipse plane is now finished).



Know: $a = \frac{3}{4}$, $a - c =$ length $\mathcal{J}$ $\#$

$$= \frac{\sqrt{3}}{2(\sqrt{3}+1)}$$

$$\Rightarrow \quad c = \frac{3}{4} - \frac{\sqrt{3}}{2(\sqrt{3}+1)}$$

and $\quad b = \sqrt{a^2 - c^2} = \sqrt{\left(\frac{3}{4}\right)^2 - \left(\frac{3}{4} - \frac{\sqrt{3}}{2(\sqrt{3}+1)}\right)^2}$

$$\boxed{\begin{array}{ll} \text{Use } \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 & \text{for ellipse} \\ \& \quad x = \dfrac{3}{4} + \dfrac{3}{2(\sqrt{3}+1)} & \text{for Directrix Line} \end{array}}$$