Useful Result

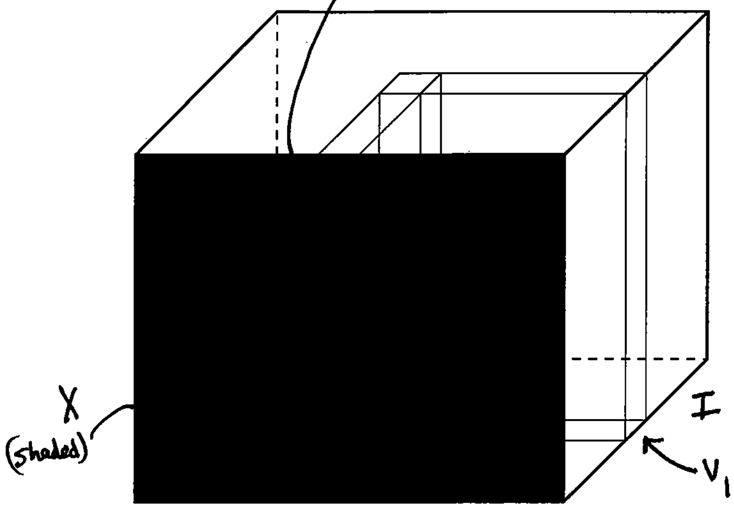
$q : X \longrightarrow Y$ a quotient map

$\Rightarrow \; q \times 1 : X \times I \longrightarrow Y \times I$

is also a quotient map

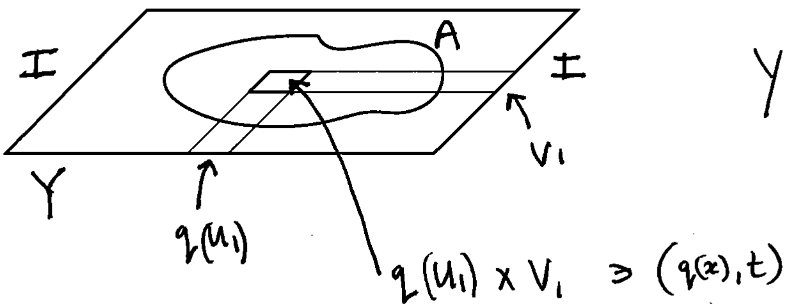
(pf works for any locally compact Hausdorff Z in place of I)

Problem: $q^{-1}(q(U_1))$ need not be open in $X \times I$.



$$\begin{array}{c} X \times I \\ \Big\downarrow q \times \mathbb{1} \\ Y \times I \end{array}$$

$q \times \mathbb{1}$ is cts & surjective, since q and $\mathbb{1}$ are.



Given: $A \subseteq Y \times I$ so that $(q \times \mathbb{1})^{-1}(A)$ is open in $X \times I$.

To show: A is open in $Y \times I$.

i.e. $\forall\ (f(x), t) \in A\ \exists$ basic open set $W \times V$ so that

$$(f(x), t) \in W \times V \subseteq A$$

<u>Intuition</u> $(x,t) \in (q \times \mathbb{1})^{-1}(A)$ is open in $X \times I$

$\Rightarrow \exists$ basic open set $U_1 \times V_1$ so that

$$(x,t) \in U_1 \times V_1 \subseteq (q \times \mathbb{1})^{-1}(A).$$

First attempt... use $q(U_1) \times V_1$ about $(q(x),t)$ in $Y \times I$.

Certainly,

$$(q(x),t) \in q(U_1) \times V_1 \subseteq A$$

(*) However, $q(U_1)$ may not be open in Y, since $q^{-1}(q(U_1))$ may not be open in X.

<u>Strategy to address (*)</u>. Enlarge U_1 to get a bigger open nbd

$$U_1 \subseteq U$$

so that $q^{-1}(q(U)) = U$.

Then $q(U)$ will be open in Y.

Needs to be done carefully — eg. Need to ensure that we remain inside $(q\times\mathbb{1})^{-1}(A)$

However . . .

(1) the enlarging procedure takes ∞ly many steps (applications of KL below).

(2) the enlarging procedure requires us to shrink V_1 slightly before we start.

$$\boxed{\begin{array}{c}\exists \text{ open nbd } V \text{ of } t \text{ so that} \\ t\in V \subseteq \overline{V} \subseteq V_1\end{array}}$$

this is compact (closed in I).

(Rk: This is guaranteed by the locally cpt Hausdorff condn in the case we replace I by a l.c. Haus. space Z.)

<u>Key Lemma</u> (KL): Let X, Y, q, A, $\overline{V}$ be as above.

Given $U \subseteq X$ open so that $U\times\overline{V} \subseteq (q\times\mathbb{1})^{-1}(A)$.

Then there exists $U'\subseteq X$ open so that

(1) $q^{-1}(q(U)) \subseteq U'$, and

(2) $U'\times\overline{V} \subseteq (q\times\mathbb{1})^{-1}(A)$.

Pf of KL : Given $z \in q^{-1}(q(U))$, for any $t \in \bar{V}$ we have

$$(z,t) \in (q \times 1\!\!1)^{-1}(A) \quad \text{which is open.}$$

$\Rightarrow \exists$ basic open set

$$(z,t) \in U_t \times V_t \subseteq (q \times 1\!\!1)^{-1}(A).$$

For a given z, the collection $\{V_t \mid t \in \bar{V}\}$ is an o.c. of compact space $\bar{V}$, $\Rightarrow \exists$ finite subcover $\{V_{t_1}, \ldots, V_{t_N}\}$.

Let $U_z = U_{t_1} \cap \cdots \cap U_{t_N}$.

$$U_z \times \bar{V} \subseteq U_z \times (V_{t_1} \cup \cdots \cup V_{t_N})$$

$$\subseteq (U_{t_1} \times V_{t_1}) \cup \cdots \cup (U_{t_N} \times V_{t_N}) \subseteq (q \times 1\!\!1)^{-1}(A).$$

i.e. For each $z \in q^{-1}(q(U))$ $\exists$ open nbd U_z so that the open tube $U_z \times \bar{V} \subseteq (q \times 1\!\!1)^{-1}(A)$

Now set $U' = \bigcup_{z \in q^{-1}(q(U))} U_z$

Check: U' is open in X,
U' contains $q^{-1}(q(U))$, and
$U' \times \bar{V} \subseteq (q \times 1\!\!1)^{-1}(A)$ ---- $\boxed{E}$ of KL

Recall what we had with our initial intuitive approach ....

$\exists$ basic open nbd $U_1 \times V$ of (x, t) in $X \times I$ so that

$$U_1 \times \overline{V} \subseteq (q \times \mathbb{1})^{-1}(A).$$

(KL) $\Rightarrow$ $\exists$ open set $U_2 \supseteq q^{-1}(q(U_1)) \supseteq U_1$ so that

$$U_2 \times \overline{V} \subseteq (q \times \mathbb{1})^{-1}(A)$$

....

inductively, (KL) $\Rightarrow$ $\exists$ open set $U_n \supseteq q^{-1}(q(U_{n-1})) \supseteq U_{n-1} \supseteq \cdots \supseteq U_1$.

so that

$$U_n \times \overline{V} \subseteq (q \times \mathbb{1})^{-1}(A).$$

Define $\boxed{U = \bigcup_{n=1}^{\infty} U_n}$

Properties of U

(easy to check)

$\rightarrow$ U is open in X

$\rightarrow$ U contains each U_n (& hence is an open nbd of x)

$\rightarrow$ $U = \bigcup_{n=2}^{\infty} U_n$

$\rightarrow$ $U \times \overline{V} \subseteq (q \times \mathbb{1})^{-1}(A)$

Crucial property of U...

$$U \subseteq q^{-1}(q(U)) = q^{-1}\left(q\left(\bigcup_{n=1}^{\infty} U_n\right)\right)$$

$$= \bigcup_{n=1}^{\infty} q^{-1}(q(U_n))$$

$$\subseteq \bigcup_{n=1}^{\infty} U_{n+1}$$

$$= \bigcup_{n=2}^{\infty} U_n = U$$

$$\Rightarrow \boxed{U = q^{-1}(q(U))}$$

$\Rightarrow$ $q(U)$ is open in Y (since $q^{-1}(q(U)) = U$ is open in X & q is a quotient map).

$$\Rightarrow (q(x), t) \in \underbrace{q(U) \times V}_{\text{basic open nbd in } Y \times I} \subseteq A.$$

$\Rightarrow$ A is open in $Y \times I$

$\Rightarrow$ $(q \times \mathbb{1})$ is a quotient map

$\blacksquare$
