MATH 2443–008 Calculus IV Spring 2014

Vector fields and Gradients in 2-dimensions

$$\left\{\begin{array}{l}\text{Functions } f(x,y)\\ \text{on a domain}\\ D \text{ in } \mathbb{R}^2.\end{array}\right\} \xrightarrow{\text{grad}} \left\{\begin{array}{l}\text{Vector fields}\\ \mathbf{F}=\langle P,Q\rangle\\ \text{on the domain}\\ D \text{ in } \mathbb{R}^2.\end{array}\right\} \xrightarrow{\text{diff}} \left\{\begin{array}{l}\text{Functions } f(x,y)\\ \text{on the domain}\\ D \text{ in } \mathbb{R}^2.\end{array}\right\}$$

1. The first operator *grad* takes a function $f = f(x,y)$ and returns the vector field

$$\text{grad}(f) \;=\; \nabla f \;=\; \langle f_x, f_y\rangle\,.$$

2. The second operator *diff* takes a vector field $\mathbf{F} = \langle P(x,y), Q(x,y)\rangle$ and returns the "difference" of partial derivatives

$$\text{diff}(\mathbf{F}) \;=\; \text{diff}(\langle P,Q\rangle) \;=\; Q_x - P_y\,.$$

Note that the partial derivatives are just functions of (x,y), and the difference of two partial derivatives is a function of (x,y).

3. It is clear that the composition of these two operators gives the zero map.

$$\text{diff}\circ\text{grad}(f) \;=\; \text{diff}(\text{grad}(f)) \;=\; \text{diff}(\langle f_x, f_y\rangle) \;=\; (f_y)_x - (f_x)_y \;=\; 0\,.$$

4. **Question.** Which vector fields $\mathbf{F} = \langle P,Q\rangle$ are the gradient vector fields of functions?

   (a) We know that one condition is that $\text{diff}(\mathbf{F}) = 0$. This gives a good test; if $\text{diff}(\mathbf{F}) \neq 0$, then we conclude that $\mathbf{F}$ is not the gradient vector field of a function.

   (b) However, we have seen that the vector field

$$\mathbf{A} \;=\; \frac{\langle -y, x\rangle}{x^2+y^2}$$

   defined on $\mathbb{R}^2 - \{(0,0)\}$ satisfies $\text{diff}(\mathbf{A}) = 0$ but $\mathbf{A}$ is not the gradient of a function.

   (c) Indeed, we saw in class that $\mathbf{A}$ is locally the gradient of the "angle function" $f(x,y) = \tan^{-1}(y/x) + c$. However, the "angle function" is not well defined on $\mathbb{R}^2 - \{(0,0)\}$; one has to add 2π every time one travels counterclockwise around a circle which encloses $(0,0)$.

   (d) Suppose $\mathbf{F}$ is a vector field defined on $\mathbb{R}^2 - \{(0,0)\}$ which satisfies: (i) $\text{diff}(\mathbf{F}) = 0$; and, (ii) $\oint_{S^1} \mathbf{F}\cdot d\mathbf{r} \neq 0$ where S^1 is the unit circle centered at $(0,0)$ with the standard counterclockwise orientation.

      i. Use Green's theorem to show that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} \;=\; \oint_{S^1} \mathbf{F}\cdot d\mathbf{r}$$

      for ANY counter clockwise oriented circle C which encircles once around $(0,0)$.

ii. Use Green's theorem to show that

$$\mathbf{F} - \frac{\oint_{S^1} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \mathbf{A}$$

is a conservative vector field. Here $\mathbf{A}$ is the special vector field introduced in (b) above.

iii. Conclude that if $\mathbf{F}$ defined on $\mathbb{R}^2 - \{(0,0)\}$ satisfies $\mathrm{diff}(\mathbf{F}) = 0$, then

$$\mathbf{F} = \nabla f + \frac{\oint_{S^1} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \mathbf{A}$$

for some scalar field (function) $f(x,y)$ on $\mathbb{R}^2 - \{(0,0)\}$. That is, $\mathbf{F}$ is a gradient plus a constant multiple of $\mathbf{A}$.

(e) More generally, suppose that $\mathbf{F}$ is a vector field defined on $\mathbb{R}^2 - \{(0,0),(p,q)\}$ which satisfies $\mathrm{diff}(\mathbf{F}) = 0$, then

$$\mathbf{F} = \nabla f + \frac{\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \frac{\langle -y, x \rangle}{x^2 + y^2} + \frac{\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}}{2\pi} \frac{\langle -(y-q), (x-p) \rangle}{(x-p)^2 + (y-q)^2}$$

for some scalar field (function) $f(x,y)$ on $\mathbb{R}^2 - \{(0,0),(p,q)\}$, and where C_1 and C_2 are small circles (bounding disjoint disks) about $(0,0)$ and (p,q) respectively. That is, $\mathbf{F}$ is a gradient plus a constant multiple of $\mathbf{A}$ and a constant multiple of $\frac{\langle -(y-q),(x-p) \rangle}{(x-p)^2+(y-q)^2}$.

(f) Generalize the result above to the case of $\mathbb{R}^2$ with N points removed.

5. **Remark.** The sets of functions and vector fields above are actually vector spaces (from your linear algebra class). Just like with regular vectors in 3-dimensions, one can add functions (or vector fields) or multiply them by constants to get new functions (or vector fields). The differential operators grad and diff respect sums and constant multiples (these are just versions of the standard "rules" of differentiation), and so are examples of *linear maps.*

(a) The *kernel* of diff is the set of all vector fields $\mathbf{F}$ defined on the domain D such that $\mathrm{diff}(\mathbf{F}) = 0$.

$$\ker(\mathrm{diff}) = \{\mathbf{F} \,|\, \mathbf{F} \text{ a vector field on } D \text{ such that } \mathrm{diff}(\mathbf{F}) = 0\}$$

This is a vector subspace of the space of vector fields.

(b) The *image* of grad is the set of all vector fields $\mathbf{F}$ of the form $\mathbf{F} = \nabla f$ for some function f defined on the domain D.

$$\mathrm{Im}(\mathrm{grad}) = \{\nabla f \,|\, f(x,y) \text{ a function on } D\}$$

This is a vector subspace of the space of vector fields.

(c) Now $\mathrm{diff} \circ \mathrm{grad} = 0$ implies that one of these spaces is a *subspace* of the other:

$$\mathrm{Im}(\mathrm{grad}) \subset \ker(\mathrm{diff})$$

Furthermore, the Green's theorem applications above tell us that the extra *dimensions* needed to pass from Im(grad) to ker(diff) is equal to the number of "holes" in the domain D.