Math 2433–006 Honors Calculus III
Tautochrone property of the cycloid

We have to show that the upside-down cycloid is a tautochrone (or isochrone); namely, a particle which starts from rest at any point of the cycloid and slides freely under gravity (with no friction) will take the same time (independent of the starting point) to reach the bottom.

Here's what we'll do.

1. We'll start with the general equation of motion (what forces are acting on the particle and what do they say about the acceleration of the particle) of a particle on a parametric curve.

2. Next, we'll integrate this to get a general equation which the physics people among you will recognize to be merely an expression of the law of conservation of energy. So, if you like, in these steps we really just proved the law of conservation of energy for the force of gravity in the context of parametric curves.

3. We take the (law of conservation of energy) equation from step 2 and solve for $\frac{dt}{ds}$ and then integrate to find the total time T taken for the particle to start at some point and to slide to the bottom of the cycloid. At this stage we'll use the fact that we're working with a cycloid, by using the parametric equations of the cycloid. The ds integral will be evaluated by changing variables to the parameter θ.

4. Evaluate the resulting $d\theta$ integral. Use half angle formulas, substitution, and some inverse trig results (feels more like Calculus II at this stage).

5. Finally we'll see what our answer for T depends on.

   - How does it depend on the radius of the original circle which defined the cycloid?
   - How does it depend on the acceleration due to gravity?
   - How does it depend on the initial position of the particle?

   Do all these answers make intuitive sense?

Step 1: Show that the equation of motion of a particle freely (no friction) sliding down a parametric curve $(x(t), y(t))$ under gravity is given by

$$\frac{d^2s}{dt^2} = -g\frac{\frac{dy}{dt}}{\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}}$$

Hint. Newton's equation says that mass times acceleration equals force (in this case the component of the force of gravity which is tangent to the curve).

Step 2: Rearrange and integrate the equation of motion in step 1 to get the law of conservation of energy equation

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 = g(y(0) - y(t))$$

Hint. Remember that the denominator in the previous equation is just $\frac{ds}{dt}$. Multiply across by this denominator, and note that the left hand side is now precisely the derivative of $\frac{1}{2}\left(\frac{ds}{dt}\right)^2$ with respect to t. Integrate with respect to t from time 0 until time t.

Step 3: Rearrange equation in step 2 to get

$$T = \int \frac{ds}{\sqrt{2g(y(0) - y(t))}}$$

Now suppose the particle starts ($t = 0$) from the point corresponding to $\theta = \alpha$ for some $0 \leq \alpha < \pi$. We want to find the time T taken to get to the bottom (corresponding to $\theta = \pi$) of the cycloid. We Now think of y as a function of the parameter θ rather than time, and so can write

$$T = \int_{\alpha}^{\pi} \frac{\frac{ds}{d\theta}\, d\theta}{\sqrt{2g(y(\alpha) - y(\theta))}}$$

Step 4: Evaluating the integral using these hints.

- Write down the parametric equations for the cycloid, and compute $\frac{ds}{d\theta}$. You should use the half angle formulas to write your answer as $\frac{ds}{d\theta} = 2R\sin(\theta/2)$.
- Also use the half angle formulas to show that

$$\frac{\cos\alpha - \cos\theta}{2} = \cos^2(\frac{\alpha}{2}) - \cos^2(\frac{\theta}{2})$$

- Substitute the results above into the integral. Finally, use the substitution $u = \frac{\cos(\theta/2)}{\cos(\alpha/2)}$ to evaluate the integral. (remember that θ is a variable and that α is a constant)

Step 5: Intuition building — interpreting the result.

- Write down your final answer for T. Does it depend on α?
- What happens to T if you move to a different planet where the acceleration due to gravity g is larger? Does this make sense?
- What happens to T when you use a larger cycloid (based on a circle with larger radius R)? Does this make sense?