MATH 5863-001 Topology II Spring 2011

Snow-break Questions

1. Give a proof that $\pi_1(T^2) = \mathbb{Z}^2$ which mirrors the proof given in class that $\pi_1(S^1) = \mathbb{Z}$. Here T^2 is the torus.

   You should use the covering space $p : \mathbb{R}^2 \to T^2$, and mimic the proof in class that $\pi_1(S^1) = \mathbb{Z}$. In particular, if $h_{(a,b)} : [0,1] \to \mathbb{R}^2$ is the constant speed, straight-line path from $(0,0)$ to (a,b), you should say why the following three paths are are all path homotopic

   $$h_{(a,b)}\,, \qquad h_{(a,0)} \cdot (T_{(a,0)} \circ h_{(0,b)})\,, \qquad h_{(0,b)} \cdot (T_{(0,b)} \circ h_{(a,0)})$$

   where $T_{(m,n)} : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (x+m, y+n)$ is the *translation by* (m,n) map. You should show that $\Psi : \mathbb{Z}^2 \to \pi_1(T^2) : (a,b) \mapsto [p \circ h_{(a,b)}]$ is an isomorphism of groups.

2. Prove that $\pi_1(S^1 \times [0,1])$ is $\mathbb{Z}$, and that $\pi_1(M^2) = \mathbb{Z}$ where M^2 denotes the Mobius band.
   [Hint: If a space deformation retracts onto S^1, then its fundamental group is isomorphic to $\pi_1(S^1)$.]

3. Let m be an integer and $f_m : \mathbb{R} \to \mathbb{R} : x \mapsto mx$. We know that f_m induces a continuous map $\widehat{f_m} : \mathbb{R}Z \to \mathbb{R}/\mathbb{Z}$ of the circle to itself. This map in turn induces a group homomorphism $\mathbb{Z} \to \mathbb{Z}$. What is this homomorphism?
   [Hint: Look at the proof that $\pi_1(S^1) = \mathbb{Z}$, and consider the effect of $(\widehat{f_m})_*$ on a generator.]

4. Prove that there is no retraction of the Mobius band to its boundary circle.
   [Hint: Think about questions 2 and 3.]

5. Prove that the wedge product of two circles (the figure 8 space, union of two circles identified along a point) has nontrivial fundamental group.
   [Hint: Use retractions to prove that $\mathbb{Z}$ is a subgroup of this group.]

6. Let X be the 2-torus minus an open disk. This is a 2-manifold with circle boundary, $\partial X = S^1$. Prove that there does not exist a retraction $X \to \partial X$.
   [Hint: Think about question 5.]