Questions about Existence of Vector and Scalar Potentials

Recall we had the following picture of the *grad, curl*, and *div* differential operators.

$$\left\{\begin{array}{l}\text{Functions}\\ f(x,y,z)\\ \text{on a}\\ \text{domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\} \xrightarrow{\text{grad}} \left\{\begin{array}{l}\text{Vector fields}\\ \mathbf{F}=\langle P,Q,R\rangle\\ \text{on the domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\} \xrightarrow{\text{curl}} \left\{\begin{array}{l}\text{Vector fields}\\ \mathbf{F}=\langle P,Q,R\rangle\\ \text{on the domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\} \xrightarrow{\text{div}} \left\{\begin{array}{l}\text{Functions}\\ f(x,y,z)\\ \text{on the}\\ \text{domain}\\ E \text{ in } \mathbb{R}^3.\end{array}\right\}$$

1. **Tests to see if a vector field has a scalar or vector potential.**
   (a) Suppose the vector field $\mathbf{F}$ is equal to ∇f for some function f (we say that $\mathbf{F}$ is conservative, and that it has a scalar potential). Then $\nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$.
   In particular, if $\mathbf{F}$ is a vector field for which $\nabla \times \mathbf{F} \neq \mathbf{0}$, then you can conclude that $\mathbf{F}$ is NOT the gradient of some function f.
   (b) Suppose the vector field $\mathbf{F}$ is equal to $\nabla \times \mathbf{G}$ for some vector field $\mathbf{G}$ (we say that $\mathbf{F}$ has a vector potential). Then $\nabla \cdot \mathbf{F} = \nabla \cdot \nabla \times \mathbf{G} = 0$.
   In particular, if $\mathbf{F}$ is a vector field for which $\nabla \cdot \mathbf{F} \neq 0$, then you can conclude that $\mathbf{F}$ is NOT the curl of some vector field $\mathbf{G}$.

2. **Suppose the vector field F satisfies $\nabla \times \mathbf{F} = \mathbf{0}$. Is it the case that F is the gradient of some function f?**
   (a) The answer can be "No." Consider the following example.

$$\mathbf{B} = \frac{\langle -y, x, 0\rangle}{x^2+y^2}$$

   - Note that the domain of $\mathbf{B}$ is all of $\mathbb{R}^3$ minus the z–axis. This domain has a *one dimensional hole*; that is, a hole which prevents the one dimensional circle

$$C: \qquad \mathbf{r}(t) = \langle \cos(t), \sin(t), 0\rangle \qquad\qquad 0 \le t \le 2\pi$$

     from being the boundary of an oriented surface contained in the domain.
   - It is easy to verify that $\nabla \times \mathbf{B} = 0$.
   - It is also easy to verify that the line integral $\oint_C \mathbf{B} \cdot d\mathbf{r} = 2\pi$.
   - Because the path integral about a closed path is non-zero, we conclude that $\mathbf{B}$ is not a gradient.
   - *Key idea: It is a global problem, not a local problem.* We saw in class notes (2–dim version) that $\mathbf{B}$ is locally the gradient of a function; for example, the polar angle function

$$f(x,y,z) = \tan^{-1}(y/x)$$

     is one such function.
     The key problem is that there is no *globally defined function* f whose gradient is $\mathbf{B}$. In particular, when one tries to extend the definition of the polar angle function above around the circle unit C in the xy–plane, it becomes multivalued (we end up being forced to conclude that values of f at some point is both α and $2\pi + \alpha$). Note that the circle C is one of the circles which is not the boundary of an oriented surface in $\mathbb{R}^3$ minus the z–axis.

(b) If the domain has *no one dimensional holes*, then every simple, closed loop C is the boundary of an oriented surface S, and then Stokes' Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} \;=\; \iint_S \mathbf{0} \cdot d\mathbf{S} \;=\; 0$$

Thus Path integrals are independent of the chosen path, and we saw in class how to use these path integrals to build a globally defined function f with $\nabla f = \mathbf{F}$. The negative of such an f is called a *(scalar) potential* for $\mathbf{F}$.

3. Suppose the vector field $\mathbf{F}$ satisfies $\nabla \cdot \mathbf{F} = 0$. Is it the case that $\mathbf{F}$ is the curl of some vector field $\mathbf{G}$?

(a) The answer can be "No." Consider the following example.

$$\mathbf{E} \;=\; \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}$$

- Note that the domain of $\mathbf{E}$ is all of $\mathbb{R}^3$ minus the origin $(0, 0, 0)$. This domain has a *two dimensional hole*; that is, a hole which prevents the two dimensional sphere S defined by $x^2 + y^2 + z^2 = 1$ from bounding a solid ball in the domain.
- It is easy to verify that $\nabla \cdot \mathbf{E} = \mathbf{0}$.
- It is also easy to verify that $\iint_S \mathbf{E} \cdot d\mathbf{S} = 4\pi$.
- Because the surface integral of $\mathbf{E}$ about the closed sphere S is non-zero, we conclude (by the result from the Stokes' Theorem handout) that $\mathbf{E}$ is not the curl of any vector field.
- *Key idea: It is a global problem, not a local problem.* Because $\nabla \cdot \mathbf{E} = 0$, it is possible to "integrate" and find locally defined vector fields $\mathbf{G}$ whose curl equals $\mathbf{E}$ (do this as an exercise; we did some examples of finding such vector fields in class).
  The problem is that there is no *globally defined vector field* $\mathbf{G}$ on $\mathbb{R}^3$ minus $(0, 0, 0)$ whose curl is $\mathbf{E}$. In particular, there is no vector field defined on all of the unit sphere $S : x^2+y^2+z^2 = 1$ whose curl is equal to $\mathbf{E}$ on S. (It is a good exercise to try extending different candidates for $\mathbf{G}$ over all of S and to think about what goes wrong.) Note that the sphere S does not bound a solid ball in $\mathbb{R}^3$ minus $(0, 0, 0)$.

(b) If the domain has *no two dimensional holes*, so that every sphere bounds a solid ball, and if $\nabla \cdot \mathbf{F} = 0$, then one can argue that $\mathbf{F}$ is the curl of another, globally defined vector field. The argument involves some integration.

A vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \mathbf{F}$ is called a *vector potential* for $\mathbf{F}$.

4. **Remark 1.** It can be shown that these are essentially the only examples that occur. Of course a space may have several one or two dimensional holes, but locally (near the holes) the examples will all look like $\mathbf{B}$ or $\mathbf{E}$.

5. **Remark 2.** The vector fields $\mathbf{B}$ and $\mathbf{E}$ are not esoteric mathematical examples. They occur in nature, and you will meet them in your physics and engineering courses.

- For example, the field $\mathbf{B}$ is (up to an appropriate positive scalar multiple) the static *magnetic field* due to a constant electric current flowing up an infinite wire along the z-axis.

- The field $\mathbf{E}$ is the standard "inverse square law, central force" field. It could be (up to an appropriate negative scalar multiple) the *gravitational field* due to a mass m at $(0, 0, 0)$. Alternatively, it could be (up to an appropriate positive/negative scalar multiple) the *electrostatic field due to a positive/negative charge* q at $(0, 0, 0)$.

6. **Remark 3. Potentials.**

   - You should check that $\nabla \cdot \mathbf{B} = 0$.
   - Verify that $\mathbf{A} = \langle 0, 0, \frac{-1}{2} \ln(x^2 + y^2) \rangle$ is a vector potential for $\mathbf{B}$; that is, $\nabla \times \mathbf{A} = \mathbf{B}$.
   - (One can verify that the domain of $\mathbf{B}$ has no two dimensional holes! It is possible to fill spheres in this domain in with solid balls in the domain.)
   - Now check that $\nabla \times \mathbf{E} = \mathbf{0}$.
   - Verify that $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a scalar potential for $\mathbf{E}$; that is, $-\nabla f = \mathbf{E}$.
   - (One can verify that the domain of $\mathbf{E}$ has no one dimensional holes; every simple, closed loop in the domain is the boundary of some oriented surface in the domain.)
   - Working with vector potentials for Magnetic fields $\mathbf{B}$ and scalar potentials for Electric fields $\mathbf{E}$ will be useful in your EM–class.