Three Versions of the Divergence Theorem

In this note we will establish versions of the Divergence Theorem which enable us to give limit formulations of div, grad, and curl.

1. At various places in this note we will be taking integrals of vector fields component by component. If $\mathbf{F} = \langle P, Q, R\rangle$ and S is a surface, then

$$\iint_S \mathbf{F}\, dS \;=\; \iint_S \langle P, Q, R\rangle\, dS \;=\; \left\langle \iint_S P\, dS, \iint_S Q\, dS, \iint_S R\, dS \right\rangle$$

There is a similar expression for triple integrals over a region E.

2. **The Divergence (Usual) Version.** The basic version of the Divergence Theorem is in your textbook. Given a vector field $\mathbf{F}$ defined on an open domain containing the region E with boundary ∂E (see your textbook for the extra assumptions made about $\mathbf{F}$ and about ∂E), then

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} \;=\; \iint_{\partial E} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \;=\; \iiint_E \nabla \cdot \mathbf{F}\, dV$$

This gives the following limit formulation of $\nabla \cdot \mathbf{F}$ (which shows that $\nabla \cdot \mathbf{F}$ is a geometric quantity; independent of the original Cartesian coordinate system definition).

$$\nabla \cdot \mathbf{F} \;=\; \lim_{\operatorname{diam}(E)\to 0} \frac{\iint_{\partial E} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS}{\operatorname{Vol}(E)} \tag{1}$$

3. **The Gradient Version.** Given a function f defined on an open domain containing E as in item 2 above, then

$$\iint_{\partial E} f \hat{\mathbf{n}}\, dS \;=\; \iiint_E \nabla f\, dV$$

As in 2 above, this gives the following limit formulation of ∇f. This shows that ∇f is a geometric quantity (independent of the original coordinate system definition).

$$\nabla f \;=\; \lim_{\operatorname{diam}(E)\to 0} \frac{\iint_{\partial E} f \hat{\mathbf{n}}\, dS}{\operatorname{Vol}(E)} \tag{2}$$

4. **The Curl Version.** Given a vector field $\mathbf{F}$ defined on an open domain containing E as in item 2 above, then

$$\iint_{\partial E} \hat{\mathbf{n}} \times \mathbf{F}\, dS \;=\; \iiint_E \nabla \times \mathbf{F}\, dV \tag{3}$$

As in 2 above, this gives the following limit formulation of $\nabla \times \mathbf{F}$. This shows that ∇f is a geometric quantity (independent of the original coordinate system definition).

$$\nabla \times \mathbf{F} \;=\; \lim_{\operatorname{diam}(E)\to 0} \frac{\iint_{\partial E} \hat{\mathbf{n}} \times \mathbf{F}\, dS}{\operatorname{Vol}(E)}$$

5. **Proof of Gradient Version.** Let $\hat{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ be the unit outward pointing normal vector to the surface ∂E. Then

$$
\begin{aligned}
\iint_{\partial E} f\hat{\mathbf{n}}\, dS &= \iint_{\partial E} \langle fn_1, fn_2, fn_3 \rangle \, dS \\
&= \left\langle \iint_{\partial E} fn_1 \, dS, \iint_{\partial E} fn_2 \, dS, \iint_{\partial E} fn_3 \, dS \right\rangle \\
&= \left\langle \iint_{\partial E} \langle f, 0, 0 \rangle \cdot \hat{\mathbf{n}} \, dS, \iint_{\partial E} \langle 0, f, 0 \rangle \cdot \hat{\mathbf{n}} \, dS, \iint_{\partial E} \langle 0, 0, f \rangle \cdot \hat{\mathbf{n}} \, dS \right\rangle \\
&= \left\langle \iiint_E \nabla \cdot \langle f, 0, 0 \rangle \, dV, \iiint_E \nabla \cdot \langle 0, f, 0 \rangle \, dV, \iiint_E \nabla \cdot \langle 0, 0, f \rangle \, dV \right\rangle \\
&= \left\langle \iiint_E f_x \, dV, \iiint_E f_y \, dV, \iiint_E f_z \, dV, \right\rangle \\
&= \iiint_E \langle f_x, f_y, f_z \rangle \, dV \\
&= \iiint_E \nabla f \, dV
\end{aligned}
$$

6. **Proof of Curl Version.** Let $\hat{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ be the unit outward pointing normal vector to the surface ∂E. Then

$$
\begin{aligned}
\iint_{\partial E} \hat{\mathbf{n}} \times \mathbf{F} \, dS &= \iint_{\partial E} \langle n_1, n_2, n_3 \rangle \times \langle F_1, F_2, F_3 \rangle \, dS \\
&= \iint_{\partial E} \langle n_2F_3 - n_3F_2, n_3F_1 - n_1F_3, n_1F_2 - n_2F_1 \rangle \, dS \\
&= \left\langle \iint_{\partial E} (n_2F_3 - n_3F_2) \, dS, \iint_{\partial E} (n_3F_1 - n_1F_3) \, dS, \iint_{\partial E} (n_1F_2 - n_2F_1) \, dS \right\rangle \\
&= \left\langle \iint_{\partial E} \langle 0, F_3, -F_2 \rangle \cdot \hat{\mathbf{n}} \, dS, \iint_{\partial E} \langle -F_3, 0, F_1 \rangle \cdot \hat{\mathbf{n}} \, dS, \iint_{\partial E} \langle F_2, -F_1, 0 \rangle \cdot \hat{\mathbf{n}} \, dS \right\rangle \\
&= \left\langle \iiint_E \nabla \cdot \langle 0, F_3, -F_2 \rangle dV, \iiint_E \nabla \cdot \langle -F_3, 0, F_1 \rangle dV, \iiint_E \nabla \cdot \langle F_2, -F_1, 0 \rangle dV \right\rangle \\
&= \left\langle \iiint_E F_{3y} - F_{2z} \, dV, \iiint_E F_{1z} - F_{3x} \, dV, \iiint_E F_{2x} - F_{1y} \, dV \right\rangle \\
&= \iiint_E \langle F_{3y} - F_{2z}, F_{1z} - F_{3x}, F_{2x} - F_{1y} \rangle \, dV \\
&= \iiint_E \nabla \times \mathbf{F} \, dV
\end{aligned}
$$

7. **Remark 1.** The proofs above are for the most part just "definition chasing". There is a key step in each proof where one applies the usual Divergence Theorem to each of the components of a vector field; the components are written as surface integrals, and the Divergence Theorem converts them into volume integrals.

8. **Intuition about div.** Let P be a point in 3-d space, and let S_r denote the sphere of radius r about P. The divergence of a vector field $\mathbf{F}$ at the point P is the (limit as $r \to 0$ of the) *net flux of* $\mathbf{F}$ *out of the sphere* S_r per unit volume.

9. **Intuition about grad.** As before, let P be a point in 3-d space, and let S_r denote the sphere of radius r about P. Then the gradient of the function f at the point P is the (limit as $r \to 0$ of the) *f–weighted vector sum of the outward pointing unit normal vectors to S_r* per unit volume.

   We already have a good intuition about grad from class notes. It is a nice exercise to see that our old intuition (for example, that ∇f at the point P points in the direction of maximum rate of increase of f at P, and is perpendicular to the level surface of f through P) makes sense from the f–weighted sum of unit normal vectors formulation. As $r \to 0$ the level surfaces of f nearby P look more and more like parallel planes cutting through the sphere S_r.

   Can you see why the f–weighted sum of unit normal vectors to S_r should end up being perpendicular to the level surface for f through P, and why it should point in the direction of increasing f?

10. **Intuition about curl.** As in the previous two cases, let P be a point in 3-d space, and let S_r denote the sphere of radius r about P. The curl of the vector field $\mathbf{F}$ at the point P is the (limit as $r \to 0$ of the) *vector sum of $\hat{\mathbf{n}} \times \mathbf{F}$ over the sphere S_r* per unit volume.

    Let's think further about what this is measuring. For example, if $\mathbf{F}$ is perpendicular to the sphere S_r at some point then $\hat{\mathbf{n}} \times \mathbf{F}$ will be zero at this point. In general, we can write $\mathbf{F}$ at a point of S_r as a sum

    $$\mathbf{F} \;=\; \mathbf{F_n} + \mathbf{F_t}$$

    of a component which is normal to S_r and a component which is tangential to S_r. Crossing with $\hat{\mathbf{n}}$ gives

    $$\hat{\mathbf{n}} \times \mathbf{F} \;=\; \hat{\mathbf{n}} \times \mathbf{F_n} + \hat{\mathbf{n}} \times \mathbf{F_t} \;=\; \hat{\mathbf{n}} \times \mathbf{F_t}$$

    So $\hat{\mathbf{n}} \times \mathbf{F}$ is only picking out the components of $\mathbf{F}$ which *flow along* the sphere S_r, or which *flow (or circulate) around* the point P. These tangential components of $\mathbf{F}$ are rotated (kept tangent to the sphere) through $\pi/2$ (the effect of crossing with $\hat{\mathbf{n}}$) and then summed over S_r.

    One way to visualize this is to think of the tangential component $\mathbf{F_t}$ vector field as a wind velocity field on a globe. Every wind vector is rotated $\pi/2$ counterclockwise and then we take the vector sum. If there is a predominant sense of wind circulation on this globe (e.g., from west to east), then the vector sum of $\hat{\mathbf{n}} \times \mathbf{F_t}$ will be perpendicular to this (in a northerly direction).

11. **Remark 2.** Q31 in section 16.9 of the course textbook asks you to derive the gradient version of the Divergence Theorem. There is a lovely application of this to *Archimedes Principle* given in Q32.

12. **Remark 3.** We can use the coordinate-free formulations of div, grad and curl to obtain intuitions about their formulas in general orthogonal curvilinear coordinate systems. Let us develop the expression for the divergence of the vector field

    $$\mathbf{F} \;=\; F_1\hat{\mathbf{u}}_1 \,+\, F_2\hat{\mathbf{u}}_2 \,+\, F_3\hat{\mathbf{u}}_3$$

    in the orthogonal curvilinear coordinate system $\mathbf{r}(u_1, u_2, u_3)$. Recall from our handout on orthogonal curvilinear coordinates, that mutually orthogonal unit vectors are defined by

    $$\hat{\mathbf{u}}_i \;=\; \frac{1}{h_i}\frac{\partial \mathbf{r}}{\partial u_i}$$

    where the h_i are the scale factors

    $$h_i \;=\; \left|\frac{\partial \mathbf{r}}{\partial u_i}\right| .$$

The component functions F_i of $\mathbf{F}$ are functions of (u_1, u_2, u_3).

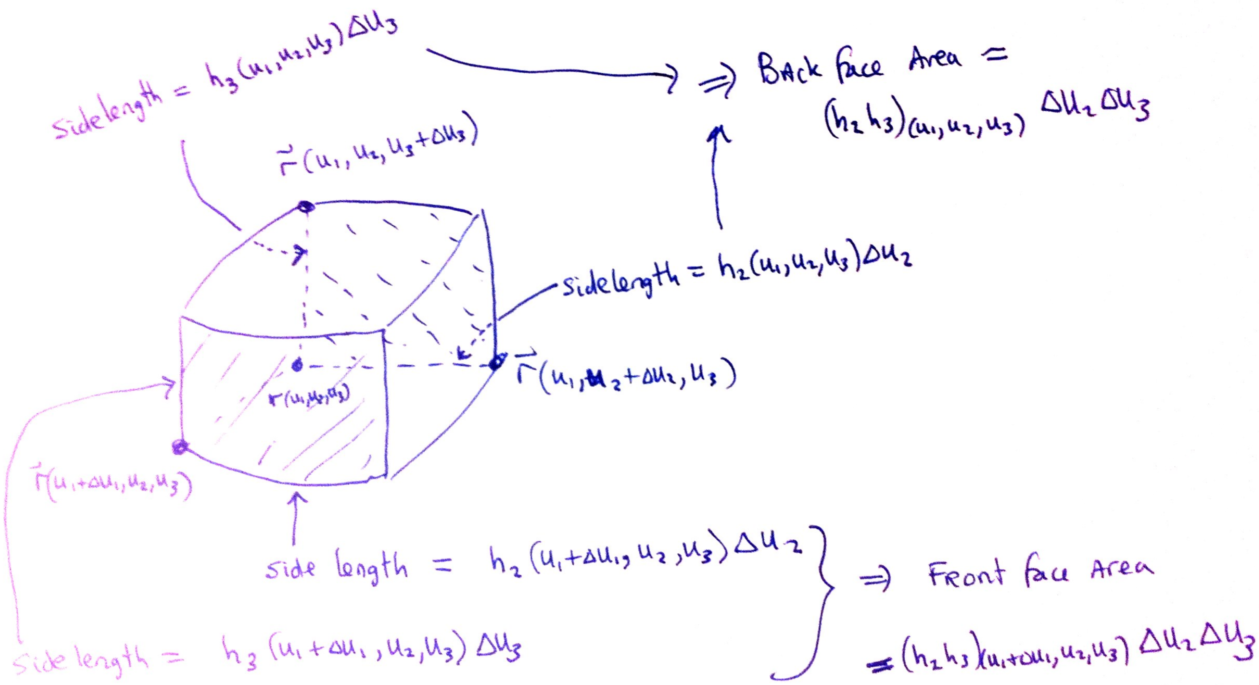
Equation (1) expresses $\nabla \cdot \mathbf{F}$ as a limit of a ratio of a surface integral to a volume

$$\nabla \cdot \mathbf{F} \;=\; \lim_{\text{diam}(E)\to 0} \frac{\iint_{\partial E} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS}{\text{Vol}(E)}$$

We will use this limit expression to compute $\nabla \cdot \mathbf{F}$ at the point $P = \mathbf{r}(u_1, u_2, u_3)$. Start with the box (cartesian product of three intervals) in parameter space

$$[u_1, u_1 + \Delta u_1] \times [u_2, u_2 + \Delta u_2] \times [u_3, u_3 + \Delta u_3]$$

This gets sent to a curvilinear box E with boundary ∂E in xyz-space. This is sketched below.



We need to compute the surface integral $\iint_{\partial E} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$. This becomes a sum of six surface integrals; one for each of the six faces of the box E. We will focus on the front and back faces which are perpendicular to the $\hat{\mathbf{u}}_1$ directions. These are shaded in the figure.

The outward pointing normal to the front face (which we denote by S_1) is $\hat{\mathbf{u}}_1|_{(u_1+\Delta u_1, u_2, u_3)}$. Likewise, the outward pointing normal vector to the back face (which we denote by S_2) is $-\hat{\mathbf{u}}_1|_{(u_1,u_2,u_3)}$. Note the sign change. Taking the dot product of these normals with the vector field $\mathbf{F}$ will select the first component F_1. Thus we get

$$\iint_{S_1} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \;\approx\; (F_1 h_2 h_3)|_{(u_1+\Delta u_1, u_2, u_3)} \Delta u_2 \Delta u_3$$

and

$$\iint_{S_2} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \;\approx\; -(F_1 h_2 h_3)|_{(u_1, u_2, u_3)} \Delta u_2 \Delta u_3$$

Now we divide by the volume of E and take a limit as $\Delta u_i \to 0$. This gives a contribution to

$$\nabla \cdot \mathbf{F} \;=\; \lim_{\text{diam}(E)\to 0} \frac{\iint_{\partial E} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS}{\text{Vol}(E)}$$

from the first two faces S_2 and S_2 of the box as follows:

$$\begin{aligned}
& \lim_{\Delta u_i \to 0} \frac{\iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS \,+\, \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS}{h_1h_2h_3\Delta u_1\Delta u_2\Delta u_3} \\
= \;& \lim_{\Delta u_i \to 0} \frac{[(F_1h_2h_3)|_{(u_1+\Delta u_1,u_2,u_3)} \,-\, (F_1h_2h_3)|_{(u_1,u_2,u_3)}]\Delta u_2\Delta u_3}{h_1h_2h_3\Delta u_1\Delta u_2\Delta u_3} \\
= \;& \lim_{\Delta u_1 \to 0} \frac{[(F_1h_2h_3)|_{(u_1+\Delta u_1,u_2,u_3)} \,-\, (F_1h_2h_3)|_{(u_1,u_2,u_3)}]}{h_1h_2h_3\Delta u_1} \\
= \;& \frac{1}{h_1h_2h_3}\frac{\partial(F_1h_2h_3)}{\partial u_1}
\end{aligned}$$

Similarly, the surface integral over the two faces with normal vectors $\hat{\mathbf{u}}_2$ gives a contribution of

$$\frac{1}{h_1h_2h_3}\frac{\partial(F_2h_1h_3)}{\partial u_2}$$

and the surface integral over the two remaining faces (with normal vectors $\hat{\mathbf{u}}_3$) gives

$$\frac{1}{h_1h_2h_3}\frac{\partial(F_3h_1h_2)}{\partial u_3}$$

Combining all three terms gives the usual formula for the divergence

$$\nabla \cdot \mathbf{F} \;=\; \frac{1}{h_1h_2h_3}\left[\frac{\partial(F_1h_2h_3)}{\partial u_1} + \frac{\partial(F_2h_1h_3)}{\partial u_2} + \frac{\partial(F_3h_1h_2)}{\partial u_3}\right]$$

We now have an intuition about this expression. The $\frac{1}{h_1h_2h_3}$ term comes from the volume of a curvilinear box; the derivative of terms like $(F_1h_2h_3)$ come from the fact that h_2h_3 gives a surface area of a curvilinear face of the box, and the partial derivative comes from taking the difference of these expressions on front and back faces of the box.

You may worry about the fact that the point $\mathbf{r}(u_1, u_2, u_3)$ was not strictly inside the box E, but was a corner point. If you wish, you may surround the point by 8 such little curvilinear boxes (obtained by replacing various $+\Delta u_i$ terms by $-\Delta u_i$ terms) and then get 8 times the expression above for the divergence. But you will be dividing by 8 times the volume too.

One can develop similar intuitions about the expressions for grad and curl. For example, the book, *Div, Grad, Curl, and All That: An Informal Text on Vector Calculus*, (Fourth Edition), by H. M. Schey, gives a friendly treatment of these ideas.