Calc IV : SP'09 Final Sol^s

Q1]... [10 points] Verify that $(0,0)$ and $(1,1)$ are critical points of the function

$$f(x,y) = 6xy - 2x^3 - 3y^2$$

Use the second derivative test to classify these two critical points.

$f_x = 6y - 6x^2$ $\qquad$ $f_y = 6x - 6y$

Critical pts (where $f_x = 0 = f_y$): $\quad x = y, \quad x^2 = y$

$\boxed{(0,0) \;\&\; (1,1)}$ ✓

$$\left.\begin{array}{l} x^2 - x = 0 \\ x(x-1) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x = 0, \ (y=0) \\ x = 1, \ (y=1) \end{array}$$

$f_{xx} = -12x \qquad f_{yy} = -6 \qquad f_{xy} = 6$

$$D = (f_{xx})(f_{yy}) - (f_{xy})^2 = (-12x)(-6) - (6)^2$$
$$= 72x - 36$$

$D(0,0) = 72(0) - 36 = -36 < 0$ $\qquad$ SADDLE AT $(0,0)$.

$$\left.\begin{array}{l} D(1,1) = 72(1) - 36 = 36 > 0 \\ f_{xx}(1,1) = -12(1) < 0 \end{array}\right\} \Rightarrow \text{LOCAL MAX AT } (1,1).$$

Q2]... [15 points] Let (x_0, y_0, z_0) be a point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Show that the tangent plane to this ellipsoid at the point (x_0, y_0, z_0) is given by the equation

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} + \frac{z_0 z}{c^2} = 1.$$

① $x_0, y_0, z_0, a, b, c, 1$ are all numbers. So the equation above is indeed the equation of a <u>plane</u>!

② $$\frac{x_0 x_0}{a^2} + \frac{y_0 y_0}{b^2} + \frac{z_0 z_0}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

because we're told (x_0, y_0, z_0) lies on the ellipsoid

$\Rightarrow$ (x_0, y_0, z_0) lies on this plane.

③ Normal vector $= \left\langle \frac{x_0}{a^2}, \frac{y_0}{b^2}, \frac{z_0}{c^2} \right\rangle$ —— (✱)

The ellipsoid is the $g = 1$ level surface of the function $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$.

∇g is normal to surface

$$\nabla g = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

$$\nabla g(x_0, y_0, z_0) = \left\langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \right\rangle$$ which is $\parallel$ to (✱).

①—③ $\Rightarrow$ plane is tangent to ellipsoid at (x_0, y_0, z_0).

Q3]...[15 points] Suppose $w = f(x, y, z)$ and $x = x(t)$, $y = y(t)$, $z = z(t)$ are all twice differentiable functions.

Write $\frac{dw}{dt}$ in terms of partial derivatives of w and derivatives of x, y, z.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \quad \text{--- Ch Rule}$$

$$= w_x x' + w_y y' + w_z z' \quad \text{--- (notation)}$$

Write $\frac{d^2w}{dt^2}$ in terms of partial derivatives of w and derivatives of x, y, z.

$$\frac{d^2w}{dt^2} = \frac{d}{dt}\left(w_x x' + w_y y' + w_z z'\right)$$

$$= \frac{d}{dt}(w_x)\, x' + w_x x'' + \frac{d}{dt}(w_y)\, y' + w_y y'' + \frac{d(w_z)}{dt} z' + w_z z''$$

--- sum + product Rules

$$= \left(w_{xx} x' + w_{xy} y' + w_{xz} z'\right) x' + w_x x''$$
$$+ \left(w_{yx} x' + w_{yy} y' + w_{yz} z'\right) y' + w_y y''$$
$$+ \left(w_{zx} x' + w_{zy} y' + w_{zz} z'\right) z' + w_z z''$$

--- Ch. Rules

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done!

Q4]... **[15 points]** The function $f(x, y, z)$ has gradient vector field given by

$$\nabla f \; = \; \langle e^{x^2}, \sin(y^2), z^2 \rangle$$

If $f(0,0,0) = 4$, find $f(0,0,3)$. [Do not try to find an explicit expression for $f(x,y,z)$. You will not be able to do the antidifferentiation.]

We know from the Fund Th^m (on attached page)

that

$$f((0,0,3)) - f((0,0,0)) = \int_C \nabla f \cdot d\vec{r}$$

where C is <u>any</u> path from (0,0,0) to (0,0,3)

$$= \int_0^3 \langle e^{0^2}, \sin(0^2), t^2 \rangle \cdot \langle 0,0,1 \rangle \, dt$$

... using straight line path C:

$\vec{r}(t) = \langle 0, 0, t \rangle$, $0 \leq t \leq 3$.

$d\vec{r} = \langle 0, 0, 1 \rangle dt$

$$= \int_0^3 t^2 \, dt$$

$$= \left. \frac{t^3}{3} \right]_0^3 = \frac{27}{3} = 9$$

$$\Rightarrow f(0,0,3) = f(0,0,0) + 9$$

$$= 4 + 9 = 13$$

Q5]... **[15 points]** Consider the parametric surface S defined by

$$\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle \quad 0 \le u \le 1, \quad o \le v \le \pi.$$

Verify that $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{1+u^2}$.

$$\vec{r}_u = \langle \cos v, \ \sin v, \ 0 \rangle$$

$$\vec{r}_v = \langle -u\sin v, \ u\cos v, \ 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle \sin v, \ -\cos v, \ u\underbrace{(\cos^2 v + \sin^2 v)}_{=1} \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + (-\cos v)^2 + u^2}$$

$$= \sqrt{1+u^2} \quad \checkmark$$

Compute the flux of the vector field $\mathbf{F} = \langle x, y, z\rangle$ over the surface S above, oriented so that the unit normal points "upward".

$$\text{Flux} = \iint_S \vec{F}\cdot\hat{n}\, dS = \int_0^{\pi}\int_0^1 \vec{F} \bullet \frac{\langle \sin v, -\cos v, u\rangle}{\cancel{\sqrt{1+u^2}}} \left(\cancel{\sqrt{1+u^2}}\, du\, dv\right)$$

$$= \int_0^{\pi}\int_0^1 \langle u\cos v, u\sin v, v\rangle \bullet \langle \sin v, -\cos v, u\rangle \, du\, dv$$

$$= \int_0^{\pi}\int_0^1 \cancel{u\cos v\sin v} - \cancel{u\sin v\cos v} + uv \quad du\, dv$$

$$= \int_0^{\pi}\int_0^1 uv\, du\, dv = \int_0^{\pi} v\, dv \int_0^1 u\, du$$

$$= \frac{v^2}{2}\Big]_0^{\pi} \ \frac{u^2}{2}\Big]_0^1$$

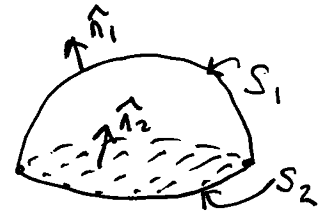
$$= \frac{\pi^2}{2}\cdot\frac{1^2}{2} = \boxed{\frac{\pi^2}{4}}$$

Q6]... [20 points] Use geometric arguments (and theorems from the course) to evaluate the following surface integrals. [You can try to compute them directly, but the geometric arguments will be much shorter].

The flux of the vector field $\mathbf{F} = \langle 0, 0, 1\rangle$ over the upper hemisphere $x^2 + y^2 + z^2 = 1,\ z \geq 0$ oriented with upward pointing unit normal.

Upper hemisphere S_1 is a surface with circle boundary (equator)

Let S_2 = equatorial disk (has same circle boundary).



over...

What we want to say is $\boxed{\iint_{S_1} \vec{F}\cdot\hat{n}_1\, dS = \iint_{S_2} \vec{F}\cdot\hat{n}_2\, dS}$ —(*)

Now the latter flux is very easy to compute

$\hat{n}_2 = \langle 0,0,1\rangle$

$\vec{F}\cdot\hat{n}_2 = 0+0+1 = 1$

$$\iint_{S_2} \vec{F}\cdot\hat{n}_2\, dS = \iint_{S_2} 1\, dS = \text{Area}(S_2) = \pi(1)^2 = \boxed{\pi}$$

The flux of the vector field $\mathbf{F} = \langle y^2, e^z, x^3\rangle$ over the surface of the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{1} = 1$$

with outward pointing unit normal.

$$\underset{\substack{\text{ellipsoid}\\ \text{surface}}}{\iint} \vec{F}\cdot\hat{n}\, dS \underset{\text{Div. Thm.}}{=} \underset{\substack{\text{solid}\\ \text{ellipsoid}}}{\iiint} \text{div}(F)\, dV$$

$$= \iiint \frac{\partial y^2}{\partial x} + \frac{\partial e^z}{\partial y} + \frac{\partial x^3}{\partial z}\, dV$$

$$= \iiint 0\, dV = 0$$

There are 2 ways to see why (*) is true . . .

① ~~Use Divergence Th^m.~~ $S_1 \cup S_2$ encloses solid $\frac{1}{2}$-ball

$$\iint_{S_1 \cup S_2} \vec{F} \cdot \hat{n} \, dS = \iiint \mathrm{div}(\vec{F}) \, dV = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 1}{\partial z} = 0$$

$= 0$

outward pointing

$$\iint_{S_1} \vec{F} \cdot \hat{n}_1 \, dS - \iint_{S_2} \vec{F} \cdot \hat{n}_2 \, dS = 0$$

outward pointing means $-\hat{n}_2 = \langle 0, 0, -1 \rangle$.

$$\Rightarrow \iint_{S_1} \vec{F} \cdot \hat{n}_1 \, dS = \iint_{S_2} \vec{F} \cdot \hat{n}_2 \, dS$$

done !

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② ~~Use stokes' Th^m.~~

$$\mathrm{curl} \langle P,Q,R \rangle = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= \langle 0, 0, 1 \rangle$$

$Q = x \quad P = 0, \ R = 0$

Let C = boundary of upper hemisphere
= equator curve

$$\mathrm{Curl}(\underbrace{\langle 0, x, 0 \rangle}_{\vec{G}}) = \langle 0, 0, 1 \rangle.$$

Now

$$\oint_C \vec{G} \cdot d\vec{r} = \iint_{S_1} \mathrm{curl}(\vec{G}) \cdot \hat{n}_1 \, dS$$

$$= \iint_{S_2} \mathrm{curl}(\vec{G}) \cdot \hat{n}_2 \, dS$$

✓

$\mathrm{Curl}(G) = \vec{F}$
& we're done!

Q7]...[10 points] Prove that the flux of the vector field $\mathbf{F} = \langle x, y, z \rangle$ over the closed surface S with outward pointing unit normal, is equal to three times the volume of the solid region enclosed by S.

$$\iint_S \langle x, y, z \rangle \cdot \hat{n} \, dS \underset{\substack{\uparrow \\ \text{div.} \\ \text{Th}^{\text{m}}}}{=} \iiint_{\text{solid}} \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dv$$

$$= \iiint_{\text{solid}} 3 \, dv$$

$$= 3 \text{ Vol}(\text{solid}).$$

Q8]. . . [20 points] Use Stokes' theorem to evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = \langle -y^3, x^3, -z^3 \rangle$, and C is the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $2x + 2y + z = 3$, oriented to agree with the orientation on the plane given by the normal vector $\langle 2, 2, 1 \rangle$.

STOKES' $\Rightarrow$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(F) \cdot \hat{n} \, dS$$

Need to parameterize S (portion of plane $2x+2y+z=3$ which is bounded by C):

$$\vec{r}(x,y) = \langle x, y, 3-2x-2y \rangle, \quad 0 \le x^2+y^2 \le 1$$

$$\vec{r}_x = \langle 1, 0, -2 \rangle$$

$$\vec{r}_y = \langle 0, 1, -2 \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle 2, 2, 1 \rangle$$

Need to compute curl(F)

$$\operatorname{Curl}(F) = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^2 \end{vmatrix}$$

$$= \left\langle 0-0, 0-0, \frac{\partial x^3}{\partial x} - \frac{\partial(-y^3)}{\partial y} \right\rangle$$

$$= \langle 0, 0, 3(x^2+y^2) \rangle$$

$$\iint_S \operatorname{curl}(F) \cdot \hat{n} \, dS = \iint_{\text{unit disk}} \langle 0, 0, 3(x^2+y^2) \rangle \cdot \frac{\langle 2,2,1 \rangle}{\cancel{|\langle 2,2,1 \rangle|}} \cancel{|\langle 2,2,1 \rangle|} \, dx\,dy$$

$$= \iint_{\text{unit disk}} 3(x^2+y^2) \, dx\,dy$$

Polar coords

$$= \int_0^{2\pi} \int_0^1 3r^2 \, r\,dr\,d\theta = 3(2\pi)\frac{1^4}{4} = \boxed{\frac{3\pi}{2}}$$

Bonus Question. Let C be a closed, non-self-intersecting, piecewise smooth curve which lies in a plane with unit normal $\mathbf{n} = \langle a, b, c\rangle$ in $\mathbf{R}^3$. Suppose that C is oriented consistently with the orientation on the plane given by $\mathbf{n}$.

Show that the area of the region in the plane enclosed by C is given by

$$\frac{1}{2}\int_C (bz - cy)dx + (cx - az)dy + (ay - bx)dz\,.$$

Stokes' Th^m tells us that this integral

$$\frac{1}{2}\int_C \langle bz-cy,\ cx-az,\ ay-bx\rangle \cdot d\vec{r}$$

is equal to

$$\frac{1}{2}\iint_S \text{curl}(\langle bz-cy,\ cx-az,\ ay-bx\rangle)\cdot \hat{n}\, dS$$

where S is the portion of the plane $ax+by+cz = d$ ← some constant.

which ~~contains~~) is bounded by C.

Parameterize S:

$$\vec{r}(x,y) = \left\langle x,\ y,\ \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y\right\rangle$$

assuming $c \neq 0$.
(otherwise use (x,z) or (y,z) as parameters)

$$\vec{r}_x = \left\langle 1, 0, -\frac{a}{c}\right\rangle$$

$$\vec{r}_y = \left\langle 0, 1, -\frac{b}{c}\right\rangle$$

$$\vec{r}_x \times \vec{r}_y = \left\langle \frac{a}{c}, \frac{b}{c}, 1\right\rangle$$

Compute Curl:

$$\text{Curl} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ bz-cy & cx-az & ay-bx \end{vmatrix}$$

$$= \langle a-(-a),\ b-(-b),\ c-(-c)\rangle$$

$$= 2\langle a, b, c\rangle$$

So $\frac{1}{2}\int_C (bz - cy)\,dx + (cx - az)\,dy + (ay - bx)\,dz$

STOKES'

$$\cancel{\tfrac{1}{2}} \iint_S \cancel{2}\langle a, b, c\rangle \cdot \frac{\langle \frac{a}{c}, \frac{b}{c}, 1\rangle}{|\langle \frac{a}{c}, \frac{b}{c}, 1\rangle|}\, dS$$

$$\parallel$$

$$= \iint_S \langle a, b, c\rangle \cdot \frac{\langle a, b, c\rangle}{|\langle a, b, c\rangle|}\, dS$$

$$= \iint_S \cancel{\frac{a^2+b^2+c^2}{\sqrt{a^2+b^2+c^2}}}^{=1}\, dS$$

since $a^2+b^2+c^2 = 1$

$\hat{n} = \langle a, b, c\rangle$

is a unit vector

(we're told this)

$$= \iint_S dS$$

$$= \text{Area}(S)$$