# Diff. & Int. Calc III — Mid I — Solutions

**Q1]... [30 points]** (a) What two pieces of geometric information do you need to uniquely specify a plane in 3–dimensional space?

① A point on the plane.

② A normal vector to the plane.

(b) Use vectors to find the equation of the plane containing the points $(1, 2, 3)$, $(1, 0, 1)$ and $(0, 2, 1)$. Show all the steps of your work.

① The difference vectors $\langle 1,2,3\rangle - \langle 0,2,1\rangle = \langle 1,0,2\rangle$ and $\langle 1,0,1\rangle - \langle 0,2,1\rangle = \langle 1,-2,0\rangle$ are both parallel to the plane.

② Therefore, their cross product $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ 1 & -2 & 0 \end{vmatrix} = \langle 4, -(-2), -2\rangle = \langle 4, 2, -2\rangle$

is normal to the plane.

③ The equation of the plane is $\langle 4,2,-2\rangle \cdot \langle x-1, y-0, z-1\rangle = 0$

$$\Rightarrow \quad 2(x-1) + (y-0) + (-1)(z-1) = 0$$

$$\Rightarrow \quad \boxed{2x + y - z - 1 = 0}$$

(c) Use vectors to find the distance from the point $(1, 1, 1)$ to the plane $x + 2y - z = 0$.

① A normal to the plane is $\vec{N} = \langle 1, 2, -1\rangle$

② $(1,0,1)$ is a point on the plane (it satisfies the equation $x+2y-z=0$).

③ The distance we seek is the component of the difference vector $\langle 1,1,1\rangle - \langle 1,0,1\rangle = \langle 0,1,0\rangle$ in the $\vec{N}$ direction.

$$\text{dist} = \left|\text{Comp}_{\vec{N}}(\langle 0,1,0\rangle)\right| = \frac{|\langle 0,1,0\rangle \cdot \langle 1,2,-1\rangle|}{|\langle 1,2,-1\rangle|}$$

$$= \boxed{\frac{2}{\sqrt{6}}}$$

Q2]... [**20 points**] a) What two pieces of geometric information do you need to uniquely specify a line in 3–dimensional space?

- A point on the line.
- A parallel vector to the line.

Find the equation of the **tangent line** to the curve $\mathbf{r}(t) = \langle t, t^2 + 1, t^3 + t \rangle$ at the point where the parameter $t = 1$. Show all the steps of your work.

① • Point $= \vec{r}(1) = \langle 1, 1+1, 1+1 \rangle = \langle 1, 2, 2 \rangle$.

② • Vector $= \left.\dfrac{d\vec{r}}{dt}\right|_{t=1} = \left.\langle 1, 2t, 3t^2+1 \rangle\right|_{t=1}$

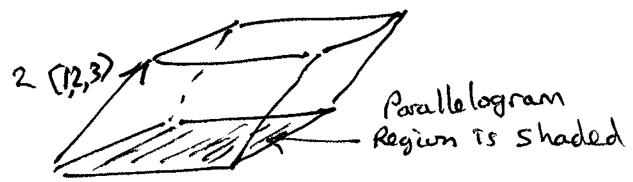
$= \langle 1, 2, 4 \rangle$

③ • Equation of tangent line is

$$\boxed{\langle x, y, t \rangle = \langle 1, 2, 2 \rangle + t\langle 1, 2, 4 \rangle \quad t \in (-\infty, \infty)}$$

**Q3]. . . [20 points]** A stream of air is flowing uniformly through space with constant velocity vector $\mathbf{v} = \langle 1, 2, 3 \rangle$ feet per second. Find the volume of air which flows through a parallelogram region with sides $\langle 1, 0, 2 \rangle$ and $\langle 2, 2, 1 \rangle$ feet in 2 seconds.

Schematic




A parallelepiped of fluid (air) flows through this region in 2 seconds. The sides of the parallelepiped are

$\rightarrow 2\langle 1, 2, 3 \rangle$ feet

$\rightarrow \langle 1, 0, 2 \rangle$ feet

$\rightarrow \langle 2, 2, 1 \rangle$ feet

Therefore, volume is

$$\left| \begin{vmatrix} 2 & 4 & 6 \\ 1 & 0 & 2 \\ 2 & 2 & 1 \end{vmatrix} \right|$$

$$= \left| 2(-4) - 4(1-4) + 6(2) \right|$$

$$= \left| -8 + 12 + 12 \right|$$

$$= 16 \text{ feet}^3.$$

**Q4]... [30 points]** (a) Consider the function of several variables

$$f(x, y) = 4x^2 - y^2$$

Draw the following **level curves**: $f = 0$; $f = 1$; $f = -1$; $f = 4$; $f = -4$.

① $f = 0$

$4x^2 - y^2 = 0$

$y^2 = 4x^2$

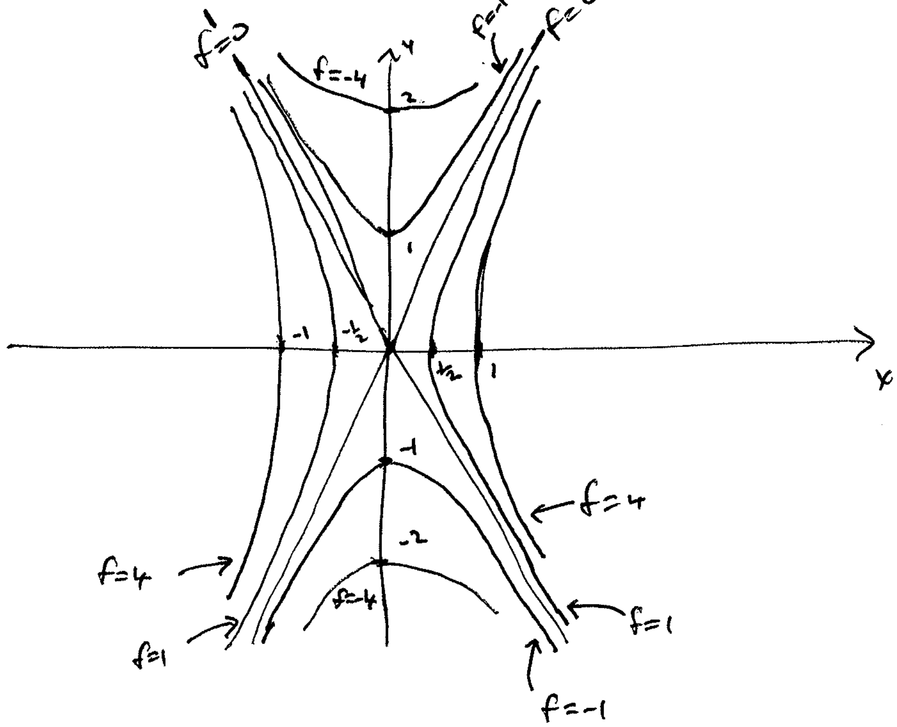
$\boxed{y = \pm 2x}$ Two lines

② $f = 1, -1, 4, -4$

are hyperbolae

which are asymptotic

to these lines.




(b) Describe the shape of the graph of the function $f(x, y) = 4x^2 - y^2$ above.

– A SADDLE SHAPE, with the 'saddle point' at $(0,0,0)$.

– Equivalently: Mountain pass

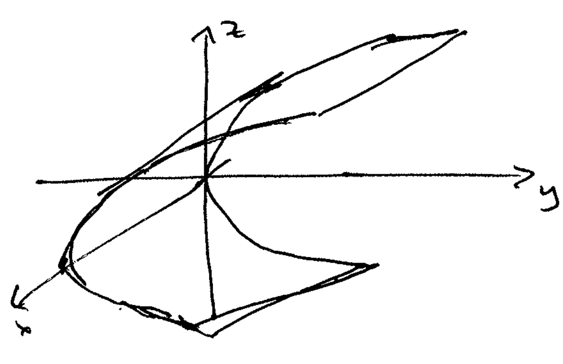
– valleys in $\pm y$ direction.

– Mountains in $\pm x$ directions.

– pass at $(0,0,0)$.

(c) Describe the surface $y = z^2$ in 3–dimensional space.

No $x$ ⟹ a cylinder. $y = z^2$ ⟹ a parabolic cylinder




Cylinder rests along $x$-axis.

**Bonus Problem.** Suppose that the point $(x_1, y_1, z_1)$ does not lie on the line $L$ given by the vector equation

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle v_1, v_2, v_3 \rangle$$

Describe the steps needed (using cross products and dot products of vectors) to find the distance from the point $(x_1, y_1, z_1)$ to the line $L$. Give details of your reasoning.

① The Line $L$ and the point $\langle x_1, y_1, z_1 \rangle$ determine a unique plane $P$ which contains them.

② The difference vector $\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ and the parallel vector $\langle v_1, v_2, v_3 \rangle$ are both parallel to the plane $P$.

③ Therefore their cross product

$$\vec{N} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \times \langle v_1, v_2, v_3 \rangle$$ is normal to $P$.

④ Therefore the cross product

$$\vec{M} = \vec{N} \times \langle v_1, v_2, v_3 \rangle$$ is (i) perpendicular to the Line $L$ and (ii) lies in the plane $P$.

⑤ The distance we want is

$$\text{dist.} = \left| \operatorname{Comp}_{\vec{M}} \left( \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \right) \right|$$

$$= \left| \frac{\vec{M} \cdot \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle}{|\vec{M}|} \right|$$