PRINT NAME: <u>SOLUTIONS</u>

Honors Calculus I [1823-001] Midterm II

For full credit, give reasons for all your answers.

Q1]...[10 points] Express each of the following limits as derivatives, $f'(a)$, of a function $f(x)$ at a point a. Hence evaluate the limits.

$$\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8}$$

If we let $f(x) = \sqrt[3]{x}$ and $a = 8$, then the limit is just

$$\begin{aligned}
\lim_{x\to 8} \frac{f(x)-f(8)}{x-8} &= f'(8) \\
&= \frac{1}{3}x^{-\frac{2}{3}}|_{x=8} \\
&= \frac{1}{3(8)^{2/3}} \\
&= \frac{1}{24}
\end{aligned}$$

$$\lim_{x\to\pi/4} \frac{2\sin(x)-\sqrt{2}}{x-\pi/4}$$

If we let $f(x) = 2\sin(x)$ and $a = \pi/4$, then the limit becomes

$$\begin{aligned}
\lim_{x\to\pi/4} \frac{f(x)-f(\pi/4)}{x-\pi/4} &= f'(\pi/4) \\
&= 2\cos(x)|_{x=\pi/4} \\
&= 2\cos(\pi/4) \\
&= 2\frac{\sqrt{2}}{2} \\
&= \sqrt{2}
\end{aligned}$$

Q2]...[10 points] Find the point on the parabola $y = x^2$ which is closest to the point $(3, 0)$ using the following geometric intuition. The line segment from $(3, 0)$ to the closest point (x, y) on the parabola will meet the parabola at right angles (i.e. it will be perpendicular to the tangent line to the parabola at the point (x, y)). Remember, when two lines are perpendicular then the product of their slopes is -1. Sketch a graph of your answer.

Any point on the parabola is of the form $(x, y) = (x, x^2)$, and so the slope of a tangent line to the parabola at this point is just $\frac{dx^2}{dx} = 2x$. In particular, the closest point that we seek is of this form.

Now, the intuition that we are told to use, is that since (x, x^2) is closest to $(3, 0)$, then the segment between them will be perpendicular. By the standard fact from coordinate geometry that is recalled in the question above, this means that the product of their slopes is -1. That is

$$\left(\frac{x^2 - 0}{x - 3}\right)(2x) \;=\; -1$$

This gives $2x^3 = 3 - x$ which rearranges to give a cubic polynomial equation as follows: $2x^3 + x - 3 = 0$. This clearly has a solution of $x = 1$ (easy to see from the coefficients: +2, +1, -3).

If you bother to divide the cubic out by $(x - 1)$ you'll get a quadratic which has no real roots, so the cubic has only got $x = 1$ a solution. Anyway, it is geometrically obvious that there would be only one closest point from the picture, and only one point on the parabola whole line segment with $(3, 0)$ is perpendicular to the parabola.

Q3]...[10 points] Find the equations of the straight lines which pass through the point $(0, 1)$ and are tangent to the graph of $y = \sqrt{2 + x^2}$.

Points of contact are of the form $(x, y) = (x, \sqrt{2 + x^2})$. The slope of the tangent line at such a point is given by $\frac{d(\sqrt{2+x^2})}{dx} = \frac{x}{\sqrt{2+x^2}}$. Since such a tangent line must also pass through the point $(0, 1)$, we have an equation which expresses the equality of the two formulas for the slope.

$$\frac{x}{\sqrt{2 + x^2}} \;=\; \frac{\sqrt{2 + x^2} - 1}{x - 0}$$

Rearranging gives $x^2 = 2 + x^2 - \sqrt{2 + x^2}$ which in turn simplifies down to $2 = \sqrt{2 + x^2}$. Squaring gives $4 = 2 + x^2$ or $x^2 = 2$ which has solutions $x = \pm\sqrt{2}$. Note that there are two points, which are symmetrically placed about the y-axis. This makes sense, since the graph is symmetric about the y-axis (even function), and the point $(0, 1)$ lies on the y-axis.

To complete the answer, we note that the slopes are $\pm\frac{\sqrt{2}}{2}$ when $x = \pm\sqrt{2}$ respectively. We know that the y-intercept is 1 since the lines contain $(0, 1)$, and so the equations are

$$y \;=\; \frac{\sqrt{2}}{2}x + 1$$

and

$$y \;=\; -\frac{\sqrt{2}}{2}x + 1\,.$$

Q4]...[15 points] Compute the derivatives y' below, stating clearly the rules that you used in each case.

$$y \;=\; \sin^4(1-x^3)$$

$$\begin{aligned} y' &= 4\sin^3(1-x^3)\cos(1-x^3)(-3x^2) && \ldots \text{ Ch. Rule twice} \\ &= -12x^2\sin^3(1-x^3)\cos(1-x^3) \end{aligned}$$

Here we also used the trig. result $\frac{d\sin x}{dx} = \cos x$, and the power and sum rules to differentiate $1-x^3$.

$$y \;=\; \frac{(x+1)(x+2)}{(x+3)(x+4)}$$

$$\begin{aligned} y' &= \frac{\frac{d[(x+1)(x+2)]}{dx}(x+3)(x+4) - (x+1)(x+2)\frac{d[(x+3)(x+4)]}{dx}}{[(x+3)(x+4)]^2} && \ldots \text{ Quotient Rule} \\ &= \frac{[(x+2)+(x+1)](x+3)(x+4) - (x+1)(x+2)[(x+4)+(x+3)]}{[(x+3)(x+4)]^2} && \ldots \text{ Product rules on numerator} \\ &= \frac{(2x+3)(x+3)(x+4) - (x+1)(x+2)(2x+7)}{[(x+3)(x+4)]^2} \end{aligned}$$

$$y \;=\; \tan(\sqrt{1+x^2}) + \cot(\sqrt{1-x^2})$$

$$\begin{aligned} y' &= \sec^2(\sqrt{1+x^2})\frac{x}{\sqrt{1+x^2}} - \csc^2(\sqrt{1-x^2})\frac{-x}{\sqrt{1-x^2}} && \ldots \text{ Sum rule, and chain rule twice on each summand} \\ &= \frac{x\sec^2(\sqrt{1+x^2})}{\sqrt{1+x^2}} + \frac{x\csc^2(\sqrt{1-x^2})}{\sqrt{1-x^2}} \end{aligned}$$

We also used the trig. results about derivatives of tan and of cot, and the power rule for the derivative of $\surd$, and of $1 \pm x^2$.

Bonus. If f is differentiable then show that

$$\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = f'(x)$$

and that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x)$$

For the first one, let $k = -h$. Note that $k \to 0$ as $h \to 0$. Thus we get

$$\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} = \lim_{k\to 0} \frac{f(x) - f(x+k)}{-k} = \lim_{k\to 0} \frac{f(x+k) - f(x)}{k}$$

This last limit is just the usual difference quotient limit, and, since we're told that $f'(x)$ exists, we know this last limit (and hence the original limit) exists and is equal to $f'(x)$.

For the second one, note that

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} &= \frac{1}{2} \lim_{h\to 0} \frac{f(x+h) - f(x) + f(x) - f(x-h)}{h} \\
&= \frac{1}{2} \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \right] \\
&= \frac{1}{2} [f'(x) + f'(x)] = f'(x)
\end{aligned}$$

The limit breaks up as a sum of the usual difference quotient limit, and the first limit above. Both of these tend to $f'(x)$. The extra 2 in the denominator cancels out the 2 from the two copies of $f'(x)$.

Show that if the first limit above exists then f must be differentiable at x, but that it is possible for the second limit to exist without $f'(x)$ existing.

If the first limit exists, then so will the usual difference quotient limit. We can see this by letting $k = -h$ as in the proof of the first part above (all steps are essentially reversible).

However, the second limit may exist without the function being differentiable (eg. $y = |x|$ at the origin) or even continuous (eg. $y = \frac{1}{x^2}$ at the origin). Here we are computing limits at the origin: $x = 0$. Both of these functions are even, so $f(0 + h) = f(h) = f(-h) = f(0 - h)$ and so the numerator is always 0. Hence the limits exist (and are 0).