MID III — SOLNS

Q1]...[10 points] Compute the following higher derivatives.

$$f^{(n)}(x) \quad \text{where} \quad f(x) = \frac{1}{2-x}$$

$f'(x) = \frac{d}{dx}(2-x)^{-1} = (-1)(2-x)^{-2}(-1) = 1(2-x)^{-2}$ $\boxed{n=1}$

$f''(x) = \frac{d}{dx}\left(1(2-x)^{-2}\right) = (1)(-2)(2-x)^{-3}(-1) = (1)(2)(2-x)^{-3}$ $\boxed{n=2}$

$f^{(n)}(x) = (1)(2)\ldots(n)(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{n+1}}$ $\boxed{\text{General Pattern}}$

$$f^{(59)}(x) \quad \text{where} \quad f(x) = \cos(3x)$$

$f'(x) = -3^1 \sin(3x)$

$f^{(2)}(x) = -3^2\cos(3x)$

$f^{(3)}(x) = 3^3 \sin(3x)$

$f^{(4)}(x) = 3^4 \cos(3x)$

Pattern.

$f^{(60)}(x) = 3^{60}\cos(3x)$

$\Rightarrow f^{(59)}(x) = 3^{59}\sin(3x)$

If $Ax^2 + By^2 = C$, then show that

$$y'' = \frac{-AC}{B^2y^3}$$

Implicit Diff$^{\underline{n}}$

$2Ax + 2By\,y' = 0$

$\Rightarrow \boxed{y' = -\dfrac{Ax}{By}}$ — (i)

Implicit again & Quotient Rule gives

$$y'' = -\left[\frac{(A)(By) - (Ax)(By')}{(By)^2}\right] = -A\left[\frac{By - Bxy'}{B^2y^2}\right] = -A\left[\frac{By^2 - Bxyy'}{B^2y^3}\right]$$ — (ii)

Final Algebra!

(i) $\Rightarrow$ $Byy' = -Ax$

$\Rightarrow -Bxyy' = +Ax^2$

Subst$^{\underline{n}}$ into (ii) gives

$y'' = -A\left[\dfrac{By^2 + Ax^2}{B^2y^3}\right] = -\dfrac{AC}{B^2y^3}$

$Ax^2+By^2=C$

Q2]... [10 points] Find the maximum and minimum values of the function $f(x) = \sin(2x) - 2\sin(x)$ on the interval $[-\pi, \pi]$. [Hint: The double angle formula $\cos(2A) = 2\cos^2(A) - 1$ may help.]

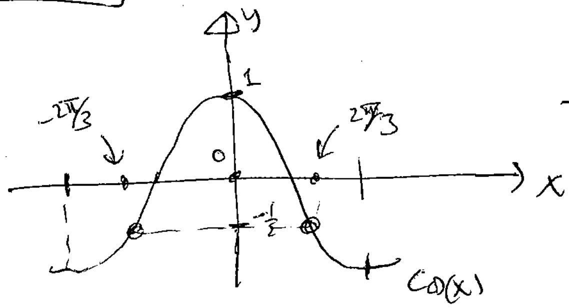
$$\begin{aligned} f'(x) &= 2\cos(2x) - 2\cos(x) \\ &= 2\left(2\cos^2(x) - 1\right) - 2\cos(x) \\ &= 2\left[2\cos^2(x) - \cos(x) - 1\right] \\ &= 2\left[2\cos(x) + 1\right]\left[\cos(x) - 1\right] \end{aligned}$$

Critical Points:

$f'(x) = 0 \iff \cos x = 1$ or $2\cos(x) + 1 = 0$

$\boxed{x = 0}$ $\qquad$ $\cos x = -\frac{1}{2}$

$\boxed{x = \pm \frac{2\pi}{3}}$

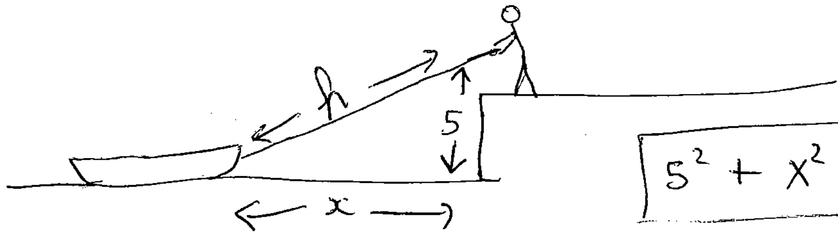


Endpoints: $-\pi, \pi$

Values:

$$\begin{aligned} f(-\pi) &= \sin(-2\pi) - 2\sin(-\pi) = 0 - 2(0) = 0 \\ f(\pi) &= \sin(2\pi) - 2\sin(\pi) = 0 - 2(0) = 0 \\ f(0) &= \sin(0) - 2\sin(0) = 0 - 2(0) = 0 \\ f\left(\frac{2\pi}{3}\right) &= \sin\left(4\pi/3\right) - 2\sin\left(2\pi/3\right) = -\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2} \leftarrow \underline{\underline{\text{MIN}}} \\ f\left(-\frac{2\pi}{3}\right) &= \sin\left(-\frac{4\pi}{3}\right) - 2\sin\left(-2\pi/3\right) = \frac{\sqrt{3}}{2} - 2\left(-\frac{\sqrt{3}}{2}\right) = +\frac{3\sqrt{3}}{2} \leftarrow \underline{\underline{\text{MAX}}} \end{aligned}$$

Q3]...[10 points] A boat is being pulled toward a pier by a rope attached to its bow. A person on the pier is pulling in the rope at a rate of 6 meters per minute. If the persons hands are 5 meters higher than the bow of the boat, how fast is the boat moving toward the pier when there are still 13 meters of rope out?



$$\boxed{5^2 + x^2 = h^2}$$ Pythagoras.

$\frac{d}{dt}$

$$0 + 2x\frac{dx}{dt} = 2h\frac{dh}{dt}$$

We're told $\frac{dh}{dt} = -6$

We're asked for $\frac{dx}{dt}$ when $h = 13$

$$x = \sqrt{(13)^2 - 5^2} = 12$$

(5, 12, 13)-TRIANGLE!

$$x\frac{dx}{dt} = h\frac{dh}{dt} \Rightarrow 12\frac{dx}{dt} = 13(-6)$$

$$\frac{dx}{dt} = -\frac{6(13)}{12}$$

$$= -\frac{13}{2} = \underline{\underline{-6\tfrac{1}{2} \text{ meters/min}}}$$

Q4]. . . [15 points] Consider the function

$$f(x) = (3 - x^2)^2$$

Find the x- and y- intercepts of f.

$$y\text{-intercept} = f(0) = (3 - 0^2)^2 = 3^2 = \boxed{9}$$

x-intercepts: solve $f(x) = 0$

$$(3 - x^2) = 0 \quad x^2 = 3 \quad \boxed{x = \pm\sqrt{3}}$$

Find the critical points of f. Determine which are local max, local min or neither. Determine the intervals where f is increasing and the intervals where f is decreasing.

$$f'(x) = 2(3 - x^2)(-2x) \qquad \text{(ch. Rule!)}$$

$$= -4x(3 - x^2)$$

Critical Points $\boxed{f'(x) = 0} \iff x = 0$ or $x^2 = 3$

$$x = \pm\sqrt{3}$$

$$\boxed{0, \pm\sqrt{3}}$$

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0, \sqrt{3})$	$(\sqrt{3}, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	↓	↑	↓	↑

$\Rightarrow$ local min at $\pm\sqrt{3}$
local Max at 0.

Determine the intervals where f is concave up and the intervals where f is concave down. Find the points of inflection of f.

$$f''(x) = -4(3 - x^2) - 4x(-2x)$$

$$= -4(3 - x^2 - 2x^2) = -12(1 - x^2) = 12(x^2 - 1)$$

$$f''(x) = 0 \quad \text{at} \quad x^2 = 1 \qquad x = \pm 1$$

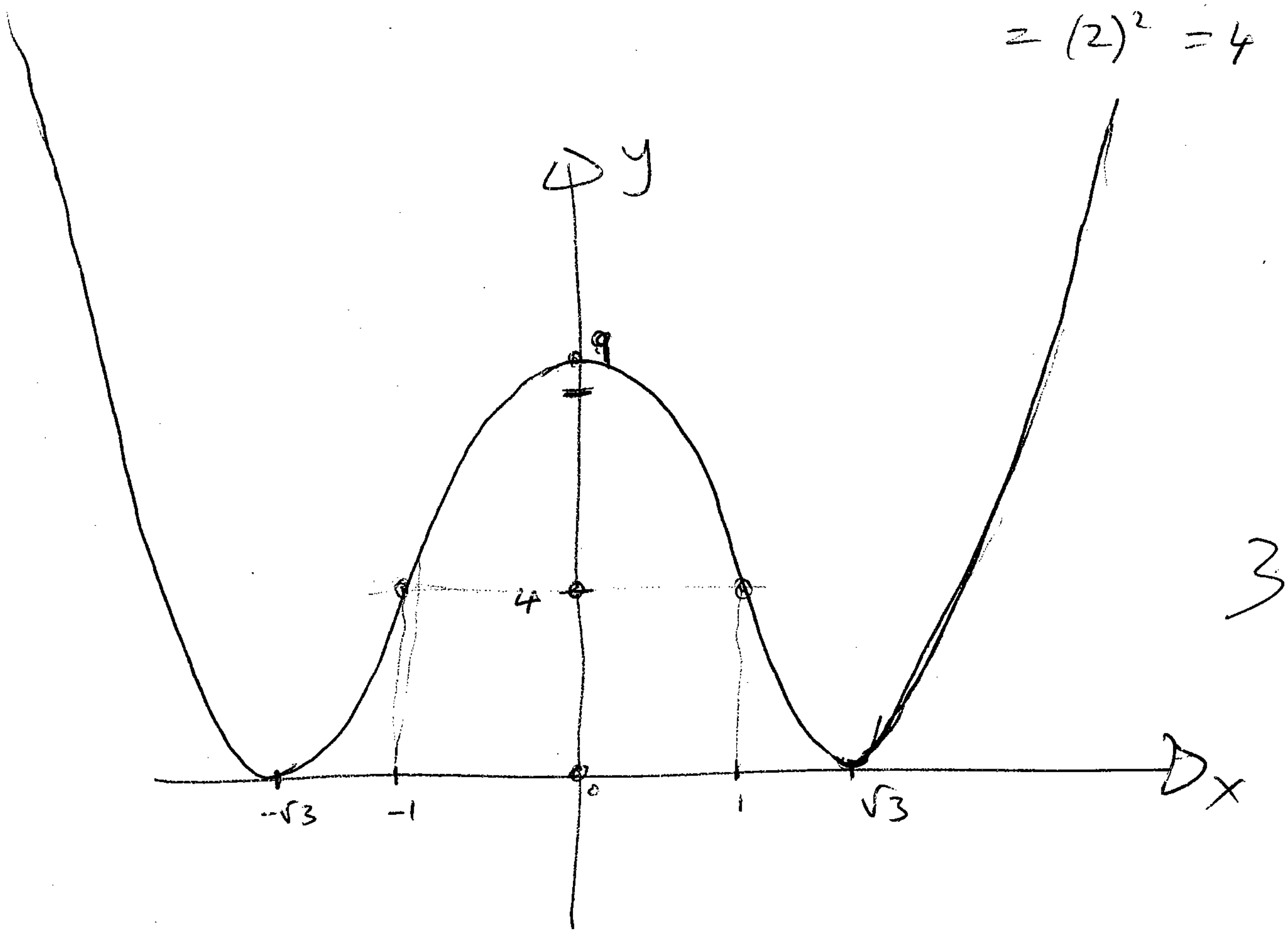
	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f''(x)$	$+$	$-$	$+$
$f(x)$	CCU	CCD	CCU

$\Rightarrow$ Inflection pts at
$x = \pm 1$.

Using the information obtained on the previous page, sketch the graph of $f(x) = (3 - x^2)^2$. Indicate the intercepts, inflection points, local maxima and local minima on your graph.

$f(\pm 1) = (3 - (\pm 1)^2)^2$

$= (2)^2 = 4$



Q5]... [5 points] Compute the linearization, $L(x)$, of the function $f(x) = \sqrt{x}$ at the point $x = 25$.

$$L(x) = f(a) + f'(a)(x-a)$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

$$f(25) = \sqrt{25} = 5$$

$$\boxed{L(x) = 5 + \frac{1}{10}(x-25)}$$

Use the linearization above to estimate the value of $\sqrt{26}$.

Approx.

$$\sqrt{26} = f(26) \approx L(26)$$

$$= 5 + \frac{1}{10}(26-25)$$

$$= 5 + \frac{1}{10}(1)$$

$$= 5.1$$

$$\boxed{\sqrt{26} \approx 5.1}$$