MID II — SOLUTIONS

Q1]...[10 points] Prove that the following are true for sets $A$and B.

$$(A\cup B)\cap\overline{(A\cap B)} = (A\cap\overline{B})\cup(B\cap\overline{A})$$

$$\begin{aligned}
(A\cup B)\cap\overline{(A\cap B)} &= (A\cup B)\cap(\overline{A}\cup\overline{B}) && \text{---- de Morgan}\\
&= (A\cap(\overline{A}\cup\overline{B}))\cup(B\cap(\overline{A}\cup\overline{B})) && \text{--- distrib } \cap \text{ over } \cup\\
&= (A\cap\overline{A})\cup(A\cap\overline{B})\cup(B\cap\overline{A})\cup(B\cap\overline{B}) && \text{.... distrib } \cap \text{ over } \cup\\
&= \emptyset\cup(A\cap\overline{B})\cup(B\cap\overline{A})\cup\emptyset && \text{---- } X\cap\overline{X}=\emptyset\\
&= (A\cap\overline{B})\cup(B\cap\overline{A}) && \text{------ } X\cup\emptyset = X
\end{aligned}$$

$$A\cup(B\setminus A) = A\cup B$$

$$\begin{aligned}
A\cup(B\setminus A) &= A\cup(B\cap\overline{A}) && \text{------ } A\setminus B = A\cap\overline{B}\\
&= (A\cup B)\cap(A\cup\overline{A}) && \text{---- distrib } \cup \text{ over } \cap\\
&= (A\cup B)\cap\mathcal{U} && \text{---- } X\cup\overline{X}=\mathcal{U}\\
&= (A\cup B) && \text{------ } X\cap\mathcal{U} = X
\end{aligned}$$

$\mathcal{U}$ = 'universe'

Q2]...[10 points] Suppose that $f : X \to Y$ is a function, and that $A \subset X$ and $B \subset X$. Prove that $f(A \cap B) \subset f(A) \cap f(B)$.

$y \in f(A \cap B) \Rightarrow y = f(x)$ for some $x \in A \cap B$

Thus $y = f(x)$ for some $x \in A$ --- since $x \in A \cap B \subseteq A$

Also $y = f(x)$ for some $x \in B$ ---- since $x \in A \cap B \subseteq B$

Thus $y \in f(A)$ and $y \in f(B)$

$\Rightarrow y \in f(A) \cap f(B)$.

$\Rightarrow \boxed{f(A \cap B) \subseteq f(A) \cap f(B)}$ — I

Give an example to show that $f(A \cap B)$ need not be equal to $f(A) \cap f(B)$.

$f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$

$A = (-\infty, 1]$ $\quad B = [-1, \infty)$ $\quad A \cap B = [-1, 1]$

$f(A) = [0, \infty)$ $\quad f(B) = [0, \infty)$ $\quad f(A \cap B) = [0, 1]$ — 2

$f(A) \cap f(B) = [0, \infty) \cap [0, \infty) = [0, \infty)$ — 1

$\boxed{[0,1] \subsetneqq [0, \infty)}$

Prove that $f(A \cap B) = f(A) \cap f(B)$ under the additional assumption that f is an injective map.

We've already seen that $f(A \cap B) \subseteq f(A) \cap f(B)$ holds in general.

Now assume f is injective.

Given $y \in f(A) \cap f(B)$. Then $y = f(a)$ for some $a \in A$ and $y = f(b)$ for some $b \in B$.

$\Rightarrow f(a) = y = f(b)$

$\Rightarrow a = b$ ---- since f is injective.

Thus $a = b \in B$ and $b = a \in A \Rightarrow a = b \in A \cap B$.

Thus $y = f(a) = f(b) \in f(A \cap B)$. Therefore $\boxed{f(A) \cap f(B) \subseteq f(A \cap B)}$ — II

Combining I & II gives equality.

Q3]... [10 points] For each of the following pairs of sets, say if they have the same cardinality or not. Give arguments (proofs) to justify your answers in each case.

$\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+$.

$\boxed{\text{Yes}}$ SAME CARDINALITY

<u>SKETCH</u> : (a) Arrange elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ as shown

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & \cdots \\
(3,1) & (3,2) & (3,3) & \cdots \\
(4,1) & (4,2) & \cdots \\
\vdots & \vdots
\end{array}$$

(b) Now list elements as shown, work through diagonals from left to right, & work down a diagonal along arrows.

(c) This <u>list</u> gives a <u>bijective</u> <u>map</u> with $\mathbb{Z}^+$

$$\begin{array}{ccccccc}
(1,1) & (1,2) & (2,1) & (1,3) & (2,2) & (3,1) & \cdots \\
\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots
\end{array}$$

More details on (b) ... Don't need to refer to "pictures"!!

(i) List elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ according to nondecreasing <u>sum</u> of coordinates

sum $= 2, 3, 4, \ldots$

(ii) Within a particular sum, $(n+1)$ say, list elements by <u>increasing</u> 1st coordinate : $(1,n), (2,n-1), (3,n-2), \ldots, (n,1)$.

$\mathbb{Z}^+$ and $(0,1) = \{x \in \mathbb{R} | 0 < x < 1\}$.

$\boxed{\text{No}}$ Different CARDINALITIES

We will prove $\nexists\ f : \mathbb{Z}^+ \to (0,1)$ which is bijective (surjective!)

Argue by contradiction. Suppose $\exists$ bijection $f : \mathbb{Z}^+ \to (0,1)$

$$\left.\begin{array}{l}
f(1) = 0.\boxed{a_{11}}\, a_{12}\, a_{13} \cdots \\
f(2) = 0.\, a_{21} \boxed{a_{22}}\, a_{23} \cdots \\
f(3) = 0.\, a_{31}\, a_{32} \boxed{a_{33}} \cdots \\
\quad \vdots \qquad\qquad\qquad \ddots
\end{array}\right] \text{ where } a_{ij} \in \{0,1,\ldots,9\} \text{ for all } i,j.$$

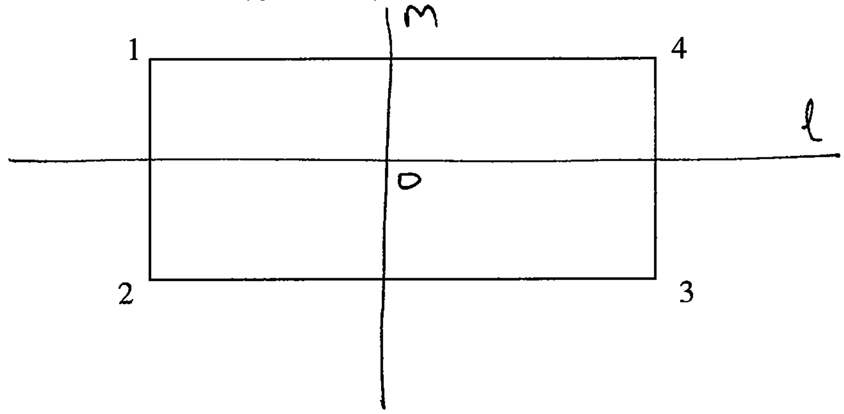
Form a new number $x = 0.b_1 b_2 b_3 \cdots \in (0,1)$ as follows.

$b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $\ldots$, $b_n \in \{0,1,\ldots,9\} \setminus \{a_{nn}\}$, $\ldots$

Clearly, $x \in (0,1)$. By construction $x \notin$ Image of f. $\Rightarrow$ f not onto. $\Rightarrow\!\Leftarrow$.

ensure $b_i \neq 0\ \forall i$, and $b_i \neq 9\ \forall i$

Q4]... [10 points] How many symmetries does the rectangle below have? Describe them, and write down a composition table for them. Also, use the vertex labeling shown to identify each symmetry with an element of the set of permutations Perm($\{1, 2, 3, 4\}$).



(Reflection in line ℓ) $= \ell \longleftrightarrow (12)(34)$

(Reflection in line m) $= m \longleftrightarrow (14)(23)$

(180° rot$^{\underline{n}}$ about O) $= R \longleftrightarrow (13)(24)$

$\mathbb{1}_{\mathbb{R}^2} \longleftrightarrow \mathbb{1} = (1)(2)(3)(4)$

There are just 4 symmetries.

Composition Table.

$\circ$	$\mathbb{1}$	R	ℓ	m
$\mathbb{1}$	$\mathbb{1}$	R	ℓ	m
R	R	$\mathbb{1}$	m	ℓ
ℓ	ℓ	m	$\mathbb{1}$	R
m	m	ℓ	R	$\mathbb{1}$