

CARDINALITY HWK SOLUTIONS

①

IMPORTANT THM : (Schröder-Bernstein)

If $f: A \rightarrow B$ is an injective map, and $g: B \rightarrow A$ is an injective map, then $\exists h: A \rightarrow B$ which is bijjective.

Q1

$$f: [0, 1) \rightarrow (0, 1)$$

$$: x \longmapsto \frac{x}{2} + \frac{1}{4}$$

$$f(x) = f(y) \Rightarrow \frac{x}{2} + \frac{1}{4} = \frac{y}{2} + \frac{1}{4}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{2}$$

$$\Rightarrow 2\left(\frac{x}{2}\right) = 2\left(\frac{y}{2}\right) \Rightarrow x = y$$

\Rightarrow f is injective

$$\text{Also } 0 \leq x < 1 \Rightarrow \frac{0}{2} \leq \frac{x}{2} < \frac{1}{2}$$

$$\Rightarrow \frac{0}{2} + \frac{1}{4} \leq \frac{x}{2} + \frac{1}{4} < \frac{1}{2} + \frac{1}{4}$$

$$\Rightarrow \frac{1}{4} \leq f(x) < \frac{3}{4}$$

We've shown: $\boxed{x \in [0, 1) \Rightarrow f(x) \in (0, 1)}$

So $f: [0, 1) \rightarrow (0, 1)$ is well-defined & injective.

②

Now $g : (0,1) \longrightarrow [0,1)$
 $: x \longmapsto x$

is clearly an injection.

Thus Schröder-Bernstein $\Rightarrow \exists$ bijection $h: (0,1) \rightarrow (0,1)$
 \Rightarrow they have same cardinality.

Q2 Consider $f : (0,1)^3 \longrightarrow (0,1)$ defined as follows.

Given $(x,y,z) \in (0,1)^3$, write each coordinate point out in decimal expansion, ensuring that you never end a decimal with an ∞ string of 9's.

This gives three unique decimals:

$$\left. \begin{array}{l} x = 0. a_1 a_2 a_3 \dots \\ y = 0. b_1 b_2 b_3 \dots \\ z = 0. c_1 c_2 c_3 \dots \end{array} \right\} a_i, b_i, c_i \in \{0, 1, \dots, 9\}$$

Now define

$$f(x,y,z) = 0. a_1 b_1 c_1 a_2 b_2 c_2 a_3 b_3 c_3 \dots \in (0,1)$$

(3)

$$f(x, y, z) = f(x', y', z')$$

$$\Rightarrow 0.a_1 b_1 c_1 a_2 b_2 c_2 \dots = 0.a'_1 b'_1 c'_1 a'_2 b'_2 c'_2 \dots$$

Since none of $0.a_1 a_2 a_3 \dots$, $0.b_1 b_2 b_3 \dots$, $0.c_1 c_2 c_3 \dots$ etc ended in an infinite string of 9's, then the same is true for the two interleaved decimal expansions above. Therefore, they ~~are~~ ^{numbers} uniquely determine their digits & so we have

$$a_1 = a'_1, b_1 = b'_1, c_1 = c'_1, a_2 = a'_2, \dots$$

$$\Rightarrow x = 0.a_1 a_2 \dots = 0.a'_1 a'_2 \dots = x'$$

$$y = 0.b_1 b_2 \dots = 0.b'_1 b'_2 \dots = y'$$

$$z = 0.c_1 c_2 \dots = 0.c'_1 c'_2 \dots = z'$$

and so $(x, y, z) = (x', y', z')$. Thus f is injective.

Note f is not bijective.

eg $0.110110100\dots$ can not equal

to $f(x, y, z)$ for any $(x, y, z) \in (0, 1)^3$,

since this would force

$$z = 0.0000\dots \notin (0, 1).$$

However, \exists injective map

$$g: (0,1) \rightarrow (0,1)^3$$

$$: x \mapsto (x, \frac{x}{2}, \frac{x}{2})$$

$$g(x) = g(x') \Rightarrow (x, \frac{x}{2}, \frac{x}{2}) = (x', \frac{x'}{2}, \frac{x'}{2})$$

$$\Rightarrow x = x'$$

so g is injective!

Finally,

Schröder-Bernstein $\Rightarrow \exists$ bijection $(0,1)^3 \rightarrow (0,1)$.

Q3

$x, y \in \mathbb{Q} \Rightarrow x, y$ have terminating or repeating pattern decimal expansions.

Even terminating \Rightarrow repeating pattern

↑
pattern of 0's

$$x = 0.a_1 \dots a_N \left(a_{N+1} \dots a_{N+k} \right) (\dots)$$

← INITIAL BLOCK → ← REPEATS →

$$y = 0.b_1 \dots b_M \left(b_{M+1} \dots b_{M+l} \right) \dots$$

← INITIAL BLOCK → ← REPEATS →

Claim: The interleaved decimal expansion

$$f(x,y) = 0.a_1 b_1 a_2 b_2 \dots$$

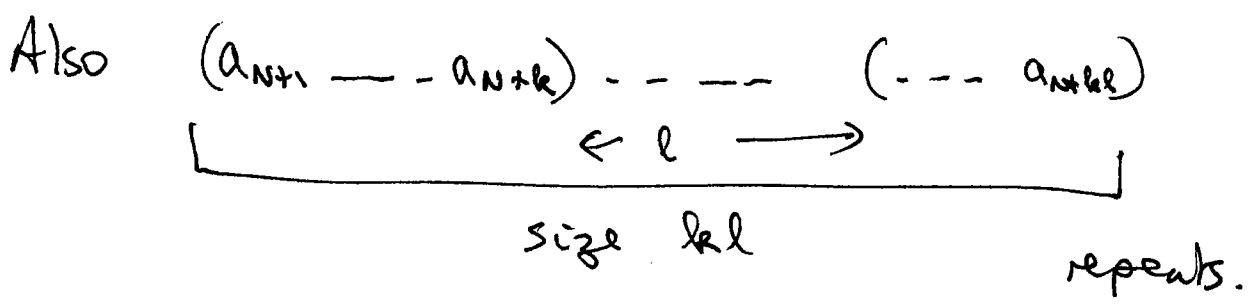
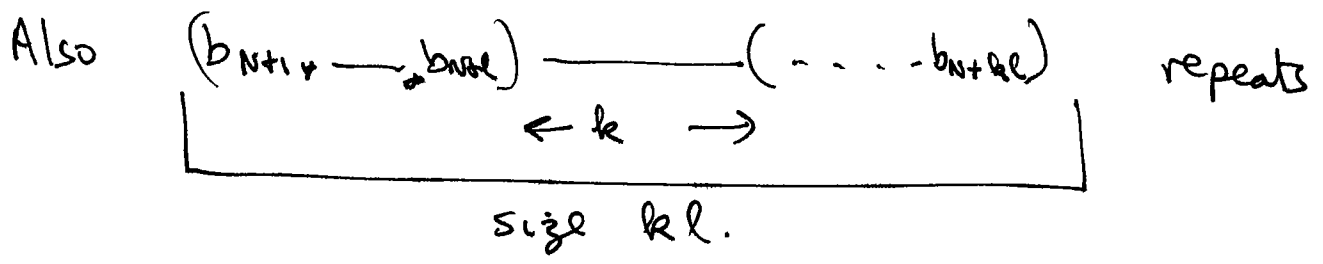
is repeating.

Thus $f(x,y) \in \mathbb{Q}$, & we're done!

Pf of claim: ~~Note that~~

Suppose $N \geq M$. Note that any block of l consecutive digits beyond b_M repeats.

In particular, $(b_{N+1} \dots b_{N+l})$ repeats.



Thus, we conclude that after position $2N$, the decimal expansion of $f(x,y)$ repeats at least every $2kl$!

Q4 Yes.

Given $f(x,y) = 0. c_1 c_2 c_3 \dots \in \mathbb{Q}$

$\Rightarrow \exists$ integers N & k so that
after c_N blocks of size k repeat.

Replace N by $N+1$ if necessary to ensure that
 N is even. Also replace k by $2k$ if
necessary to ensure that k is even.

Here is the situation

$$f(x,y) = 0. c_1 c_2 \dots c_N \overbrace{(c_{N+1} \dots c_{N+k})}^{\text{Repeats}}$$

$\leftarrow \text{Even} \rightarrow$ $\leftarrow \text{Even} \rightarrow$

Thus $x = 0. c_1 c_2 \dots c_{N-1} \overbrace{(c_{N+1} \dots c_{N+k-1})}^{\text{Repeats}}$

$\leftarrow N/2 \rightarrow$ $\leftarrow k/2 \rightarrow$

& $y = 0. c_2 c_4 \dots c_N \overbrace{(c_{N+2} \dots c_{N+k})}^{\text{Repeats}}$

$\leftarrow N/2 \rightarrow$ $\leftarrow k/2 \rightarrow$

$\Rightarrow x, y \in \mathbb{Q}$



