# CARDINALITY HWK SOLUTIONS

<u>IMPORTANT THM</u>: (Schröder-Bernstein)

If $f: A \to B$ is an injective map, and $g: B \to A$ is an injective map, then $\exists\ h: A \to B$ which is <u>bijective</u>.

---

## Q1

$$f: [0,1) \longrightarrow (0,1)$$
$$: x \longmapsto \frac{x}{2} + \frac{1}{4}$$

$$f(x) = f(y) \Rightarrow \frac{x}{2} + \frac{1}{4} = \frac{y}{2} + \frac{1}{4} \quad (-1/4 \text{ on both sides})$$
$$\Rightarrow \frac{x}{2} = \frac{y}{2}$$
$$\Rightarrow 2\left(\frac{x}{2}\right) = 2\left(\frac{y}{2}\right) \Rightarrow x = y$$

$\Rightarrow f$ is injective

Also $0 \leq x < 1 \Rightarrow \frac{0}{2} \leq \frac{x}{2} < \frac{1}{2}$

$$\Rightarrow \frac{0}{2} + \frac{1}{4} \leq \frac{x}{2} + \frac{1}{4} < \frac{1}{2} + \frac{1}{4}$$
$$\Rightarrow \frac{1}{4} \leq f(x) < \frac{3}{4}$$

We've shown: $\boxed{x \in [0,1) \Rightarrow f(x) \in (0,1)}$

So $f: [0,1) \longrightarrow (0,1)$ is <u>well-defined</u> & <u>injective</u>.

Now $g : (0,1) \longrightarrow [0,1)$

$: x \longmapsto x$

is clearly an injection.

Thus Schröder - Bernstein $\Rightarrow$ $\exists$ bijection $h : [0,1) \to (0,1)$

$\Rightarrow$ they have same cardinality.

---

**Q2** Consider $f : (0,1)^3 \longrightarrow (0,1)$ defined as follows.

Given $(x, y, z) \in (0,1)^3$, write each coordinate point out in decimal expansion, ensuring that you never end a decimal with an $\infty$ string of 9's.

This gives three unique decimals:

$$\left.\begin{aligned} x &= 0.\,a_1\, a_2\, a_3 \,\text{----} \\ y &= 0.\,b_1\, b_2\, b_3 \,\text{---} \\ z &= 0.\,c_1\, c_2\, c_3 \,\text{---} \end{aligned}\right\} \quad a_i, b_i, c_i \in \{0, 1, \ldots, 9\}$$

Now define

$$f(x, y, z) = 0.\,a_1 b_1 c_1 a_2 b_2 c_2 a_3 b_3 c_3 \,\text{-----}$$

$$\in (0,1)$$

$$f(x,y,z) = f(x',y',z')$$

$$\Rightarrow 0.a_1 b_1 c_1 a_2 b_2 c_2 \cdots = 0.a_1' b_1' c_1' a_2' b_2' c_2' \cdots$$

Since none of $0.a_1a_2a_3\cdots$, $0.b_1b_2b_3\cdots$, $0.c_1c_2c_3\cdots$ etc ended in an infinite string of 9's, then the same is true for the two interleaved decimal expansions above. Therefore, the numbers uniquely determine their digits & so we have

$$a_1 = a_1',\ b_1 = b_1',\ c_1 = c_1',\ a_2 = a_2',\ \cdots$$

$$\Rightarrow x = 0.a_1a_2\cdots = 0.a_1'a_2'\cdots = x'$$
$$y = 0.b_1b_2\cdots = 0.b_1'b_2'\cdots = y'$$
$$z = 0.c_1c_2\cdots = 0.c_1'c_2'\cdots = z'$$

and so $(x,y,z) = (x',y',z')$. Thus $f$ is injective.

---

Note $f$ is <u>not</u> bijective.

eg $0.110110100\cdots$ can <u>not</u> equal to $f(x,y,z)$ for any $(x,y,z) \in (0,1)^3$, since this would force

$$z = 0.0000\cdots \notin (0,1).$$

---

However, $\exists$ injective map

$$g : (0,1) \longrightarrow (0,1)^3$$
$$: x \longmapsto \left(x, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$g(x) = g(x') \Rightarrow \left(x, \tfrac{1}{2}, \tfrac{1}{2}\right) = \left(x', \tfrac{1}{2}, \tfrac{1}{2}\right)$$
$$\Rightarrow x = x'$$

so $g$ is injective!

---

Finally,
Schröder - Bernstein $\Rightarrow \exists$ bijection $(0,1)^3 \rightarrow (0,1)$.

---

**Q3** $x, y \in \mathbb{Q} \Rightarrow x, y$ have terminating or repeating pattern decimal expansions.

Even terminating $\Rightarrow$ repeating pattern

pattern of 0's

INITIAL BLOCK $\qquad$ $\leftarrow$ REPEATS $\rightarrow$

$$x = 0.a_1 \text{ - - - - } a_N \left(a_{N+1} \text{ —— } a_{N+k}\right)\left(\text{- - - -}\right)$$

$$y = 0.b_1 \text{ - - - } b_M \left(b_{M+1} \text{ - - - } b_{M+l}\right) \text{ - - - -}$$

$\leftarrow$ INITIAL BLOCK $\rightarrow$ $\qquad$ Repeats $\rightarrow$

Claim: The interleaved decimal expansion

$$f(x,y) = 0.\,a_1 b_1 a_2 b_2 \cdots\cdots$$

is repeating.

Thus $f(x,y) \in \mathbb{Q}$, & we're done!

---

Pf of claim: ~~(Note that)~~

Suppose $N \geq M$. Note that any block of $l$ consecutive digits beyond $b_M$ repeats.

In particular, $(b_{N+1} \text{ --- } b_{N+l})$ repeats.

Also $\underbrace{(b_{N+1} \text{ --- } b_{N+l}) \text{ ——— } (\cdots b_{N+kl})}_{\text{size } kl}$ repeats

$\leftarrow k \rightarrow$

Also $\underbrace{(a_{N+1} \text{ --- } a_{N+k}) \text{ - - - } (\cdots a_{N+kl})}_{\text{size } kl}$ repeats.

$\leftarrow l \rightarrow$

Thus, we conclude that after position $2N$, the decimal expansion of $f(x,y)$ repeats at least every $2kl$!

**Q4** Yes.

Given $f(x,y) = 0.c_1c_2c_3 \cdots\cdots \in \mathbb{Q}$

$\Rightarrow \exists$ integers $N$ & $k$ so that after $c_N$ blocks of size $k$ repeat.

Replace $N$ by $N+1$ if necessary to ensure that $N$ is <u>even</u>. Also replace $k$ by $2k$ if necessary to ensure that $k$ is even.

Here is the situation ....

$$f(x,y) = \underbrace{0.c_1c_2 \cdots c_N}_{\text{Even}} \overbrace{\underbrace{(c_{N+1} \cdots c_{N+k})}_{\text{even}}}^{\text{Repeats}}$$

Thus $$x = 0.\underbrace{c_1c_3 \cdots c_{N-1}}_{N/2} \underbrace{(c_{N+1} \cdots c_{N+k-1})}_{\substack{k/2 \\ \text{Repeats.}}}$$

& $$y = 0.\underbrace{c_2c_4 \cdots c_N}_{N/2} \underbrace{(c_{N+2} \cdots c_{N+k})}_{\substack{k/2 \\ \text{Repeats.}}}$$

$\Rightarrow x, y \in \mathbb{Q}$.

$\boxed{3}$