MID III — SOLUTIONS —

Q1]... **[10 points]** Find the absolute maximum and absolute minimum values of the function

$$f(x) = x^3 - 3x^2 - 9x + 2$$

on the interval $[-2, 4]$.

Recall there are 3 steps.

① Endpoints. -2 & 4

② Critical points.

$$f'(x) = 3x^2 - 6x - 9$$
$$= 3(x^2 - 2x - 3)$$
$$= 3(x-3)(x+1)$$

$f'(x) = 0$ when $x=3$ & $x=-1$

③ List Outputs, choose max/min.

$$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 2 = -8 - 12 + 18 + 2 = 0$$
$$f(4) = (4)^3 - 3(4)^2 - 9(4) + 2 = 64 - 48 - 36 + 2 = -18$$
$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 2 = -1 - 3 + 9 + 2 = 7$$
$$f(3) = 3^3 - 3(3)^2 - 9(3) + 2 = 27 - 27 - 27 + 2 = -25$$

Max. value is 7, occurs at $x = -1$.

Min. value is -25, occurs at $x = 3$.

Q2]...[10 points] Find the minimum distance from the point $(3,1)$ to the graph of the function $y = x^2 + 1$. What are the coordinates of the point on the graph where this minimum distance is achieved?

Minimize $\quad (\text{dist})^2 = (x-3)^2 + (y-1)^2$

$$= (x-3)^2 + \left((x^2 \cancel{+1}) \cancel{-1}\right)^2$$

$$= x^4 + x^2 - 6x + 9$$

$$f(x) = x^4 + x^2 - 6x + 9$$

$$f'(x) = 4x^3 + 2x - 6$$

$$f'(1) = 4 + 2 - 6 = 0 \qquad \left(1 \text{ is a critical point}\right)$$

Note • $f''(x) = 12x^2 + 2 \geq 2 > 0 \Rightarrow$ Local min at 1.

• $f''(x) > 0 \Rightarrow f''$ Never $= 0$

$\Rightarrow f'(x)$ has only one real Root (at $x = 1$)

(two Roots $\Rightarrow f''(x) = 0$ at some point by Rolle's Th$^{\underline{m}}$)

This does <u>NOT</u> happen.

So Minimum is achieved at $x = 1 \Rightarrow y = 1^2 + 1 = 2$

$\Rightarrow$ the point $(1,2)$ is the closest point on the graph to $(3,1)$.

Q3]. . . [20 points] Consider the function

$$f(x) = \frac{-x}{x^2+4}$$

Compute $f'(x)$.

$$f'(x) = \frac{(-1)(x^2+4)-(-x)(2x)}{(x^2+4)^2} = \frac{-x^2-4+2x^2}{(x^2+4)^2} = \frac{x^2-4}{(x^2+4)^2}$$

Find the intervals where $f(x)$ is increasing, and the intervals where $f(x)$ is decreasing. Identify (compute) any local maxima and local minima.

Critical pts, $f'(x)=0$. Only when numerator $=0$, $x^2-4=0$

$x^2=4$

$x=\pm 2$

Test Intervals	$(-\infty,-2)$	$(-2,2)$	$(2,\infty)$
x^2-4 $\boxed{\text{Denom. } \oplus}$	+	−	+
$f(x)$	↑	↓	↑

Local Max −2

Local Min at 2

Compute $f''(x)$.

$$f''(x) = \frac{(2x)(x^2+4)^2-(x^2-4)2(x^2+4)'(2x)}{((x^2+4)^2)^2} = \frac{2x(x^2+4)[x^2+4 \quad -2x^2 \quad +8]}{(x^2+4)^4} = \frac{2x[12-x^2]}{(x^2+4)^3}$$

Find the intervals where $f(x)$ is concave up, and the intervals where $f(x)$ is concave down. Identify (compute) any inflection points.

$f''(x)=0$ only when numerator $=0$ $\quad 2x[12-x^2]=0$

$x=0,\ x=\pm\sqrt{12}$

Test intervals	$(-\infty,-\sqrt{12})$	$(-\sqrt{12},0)$	$(0,\sqrt{12})$	$(\sqrt{12},\infty)$
$2x(12-x^2)$ $\boxed{\text{Denom. } \oplus}$	+	−	+	−
$f(x)$	CCU	CCD	CCU	CCD

$-\sqrt{12}, 0, \sqrt{12}$ are all inflection points.

Q3 continued...

Determine the behavior of $f(x)$ as x tends to ∞ and to $-\infty$. Are there any asymptotes?

• Denom $\neq 0 \Rightarrow$ No vertical Asymptotes

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\frac{-x}{x^2}}{\frac{x^2}{x^2}+\frac{4}{x^2}} = \frac{0}{1+0} = 0$$

Thus $y=0$ is horiz Asymptote

Determine any x- and y-intercepts that $f(x)$ has.

$\boxed{y\text{-int}} = f(0) = 0$

x-int when $f(x)=0$, when numerator $=0$, ie. $-x=0$

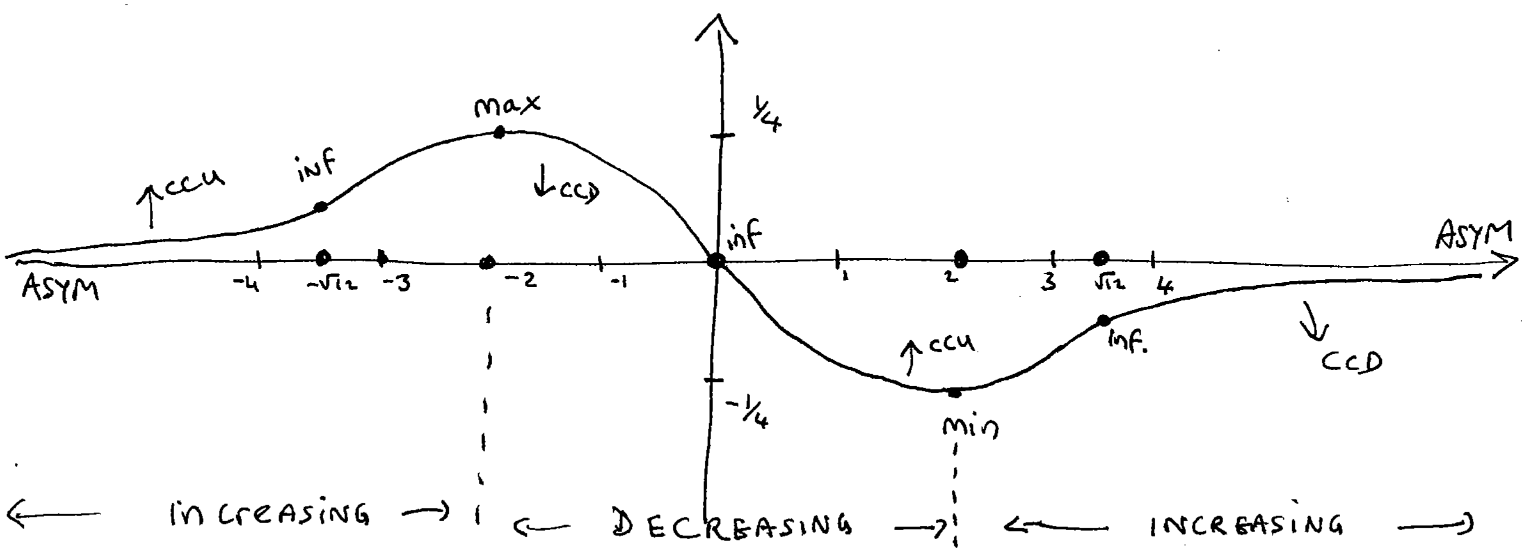
$x=0$

Draw the graph of $f(x)$, indicating all the information above (intercepts, max/min, inflection points, asymptotes) on the graph.

Note $f(-2) = \dfrac{-(-2)}{(-2)^2+4} = \dfrac{2}{8} = \dfrac{1}{4}$

$f(2) = -\frac{1}{4}$

Also note $f(x)$ is odd (symm about origin).



inf = inflection

Max = local Maximum

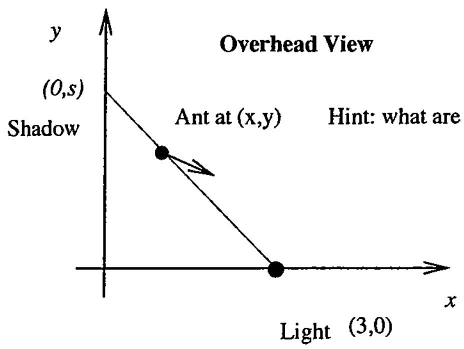
Min = local Minimum

CCU = concave up

CCD = concave down.

ASYM = ASYMPTOTE

Q4]...[10 points] A lamp is located at the point $(3,0)$ in the xy-plane. An ant is crawling along in the first quadrant of the xy-plane, and the lamp casts its shadow onto the y-axis. How fast is the ant's shadow moving along the y-axis when the ant is at the point $(1,2)$, and is moving so that its x-coordinate is increasing at a rate of $1/3$ units/second and its y-coordinate is decreasing at a rate of $1/4$ units/second?



Hint: what are you told about dx/dt and dy/dt?

We're told $\frac{dx}{dt} = \frac{1}{3}$

& $\frac{dy}{dt} = -\frac{1}{4}$

Ant at (x,y) & shadow at $(0,s)$. Both points make same slope line segments with the light (at $(3,0)$). Thus

$$\frac{y-0}{x-3} = \frac{s-0}{0-3}$$

(s, x, y all vary with respect to time, t.)

$$\Rightarrow \quad s = \frac{-3y}{x-3}$$

Take $\frac{d}{dt}$ of both sides

$$\Rightarrow \quad \frac{ds}{dt} = \frac{d}{dt}\left(\frac{-3y}{x-3}\right)$$

$\frac{d(x-3)}{dt} = \frac{dx}{dt} - 0 = \frac{dx}{dt}$

$$\underset{\text{Q.R.}}{=} \frac{-3\frac{dy}{dt}(x-3) - (-3y)\frac{dx}{dt}}{(x-3)^2}$$

Now let $x = 1$, $y = 2$, $\frac{dx}{dt} = \frac{1}{3}$, $\frac{dy}{dt} = -\frac{1}{4}$ & we get

$$\frac{ds}{dt} = \frac{(-3)(-\frac{1}{4})(1-3) - (-3(2))\frac{1}{3}}{(1-3)^2} = \frac{-\frac{3}{2} + 2}{4}$$

$$= \boxed{\frac{1}{8} \text{ units/second}}$$