UNIV 1000 : MID II SOLUTIONS

Q1]...[10 points] Differentiate the following function

$$f(x) = \tan\left(\sqrt{x^2+4x+1}\right)$$

$$f'(x) = \frac{d}{dx}\left(\tan\left(\sqrt{x^2+4x+1}\right)\right)$$

$$= \sec^2\left(\sqrt{x^2+4x+1}\right) \cdot \frac{d}{dx}\left(\sqrt{x^2+4x+1}\right) \quad \text{--- Ch. Rule}$$

$$= \sec^2\left(\sqrt{x^2+4x+1}\right) \cdot \frac{1}{2\sqrt{x^2+4x+1}} \cdot \frac{d}{dx}\left(x^2+4x+1\right) \quad \text{--- Ch. Rule again}$$

$$= \sec^2\left(\sqrt{x^2+4x+1}\right) \cdot \frac{1}{2\sqrt{x^2+4x+1}} \cdot (2x+4)$$

Suppose $f''(x)$ exists at all points x of an interval I. If f vanishes at three distinct points of I, show that f'' must vanish at some point of I.

Label the points x_1, x_2, x_3 in ascending order;

$$x_1 < x_2 < x_3 .$$

$f(x_1) = 0 = f(x_2)$. Rolle's Th^m $\Rightarrow$ there is a point c_1 in (x_1, x_2) so that $f'(c_1) = 0$

$f(x_2) = 0 = f(x_3)$. Rolle's Th^m $\Rightarrow$ there is a point c_2 in (x_2, x_3) so that $f'(c_2) = 0$.

Now look at $f'(x)$ on the interval $[c_1, c_2]$.

$f'(c_1) = 0 = f'(c_2)$. Rolle's $\Rightarrow$ there is a point c in (c_1, c_2) so that

$$(f')'(c) = 0 .$$

Thus $f''(c) = 0$.

Q2]... [20 points] Sketch the graph of the function

$$f(x) = x^{1/3}(x-4)$$

after you have answered the following questions. Make sure that your answers to these questions are visible/highlighted on your graph.

1. Find the intercepts of $y = f(x)$ and determine the behavior of $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

x-int: $x^{\frac{1}{3}}(x-4) = 0$

$x^{\frac{1}{3}} = 0$ | $x-4=0$

$x=0$ | $x=4$

X-intercepts

$f(0) = 0 \leftarrow$ y-intercept

$\lim_{x\to\infty}(x-4) = \infty$, $\lim_{x\to\infty} x^{\frac{1}{3}} = \infty$

Thus $\lim_{x\to\infty} x^{\frac{1}{3}}(x-4) = \infty$.

$\lim_{x\to-\infty}(x-4) = -\infty$, $\lim_{x\to-\infty} x^{\frac{1}{3}} = -\infty$

Thus $\lim_{x\to-\infty} x^{\frac{1}{3}}(x-4) = +\infty$.

2. Compute the derivative $f'(x)$, and find all the critical points of $f(x)$.

$f(x) = x^{4/3} - 4x^{1/3}$

$f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \frac{4}{3}x^{-2/3}$

$= \frac{4x-4}{3x^{2/3}} = \frac{4(x-1)}{3x^{2/3}}$

Critical points are where $f'(x)$ DNE

$x=0$

and where $f'(x)=0$

$x=1$

Horizontal Tangent Line at $x=1$.

Vertical Tangent Line at $(0,0)$.

3. Determine the intervals where $f(s)$ is increasing, and where $f(x)$ is decreasing.

	$(-\infty, 0)$	$(0,1)$	$(1,\infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\downarrow$	$\downarrow$	$\uparrow$

Note: $\frac{4}{3x^{2/3}}$ is $+$ for $x \neq 0$

Thus Look at $(x-1)$ signs.

Local min at $x=1$, $f(1) = 1^{\frac{1}{3}}(1-4) = -3$

4. Compute $f''(x)$, and determine the intervals where $f(x)$ is CCU, and where $f(x)$ is CCD. Does the graph of $f(x)$ have inflection points?

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3}$$

$$= \frac{4x+8}{9x^{5/3}}$$

$$= \frac{2(x+2)}{9x^{5/3}}$$

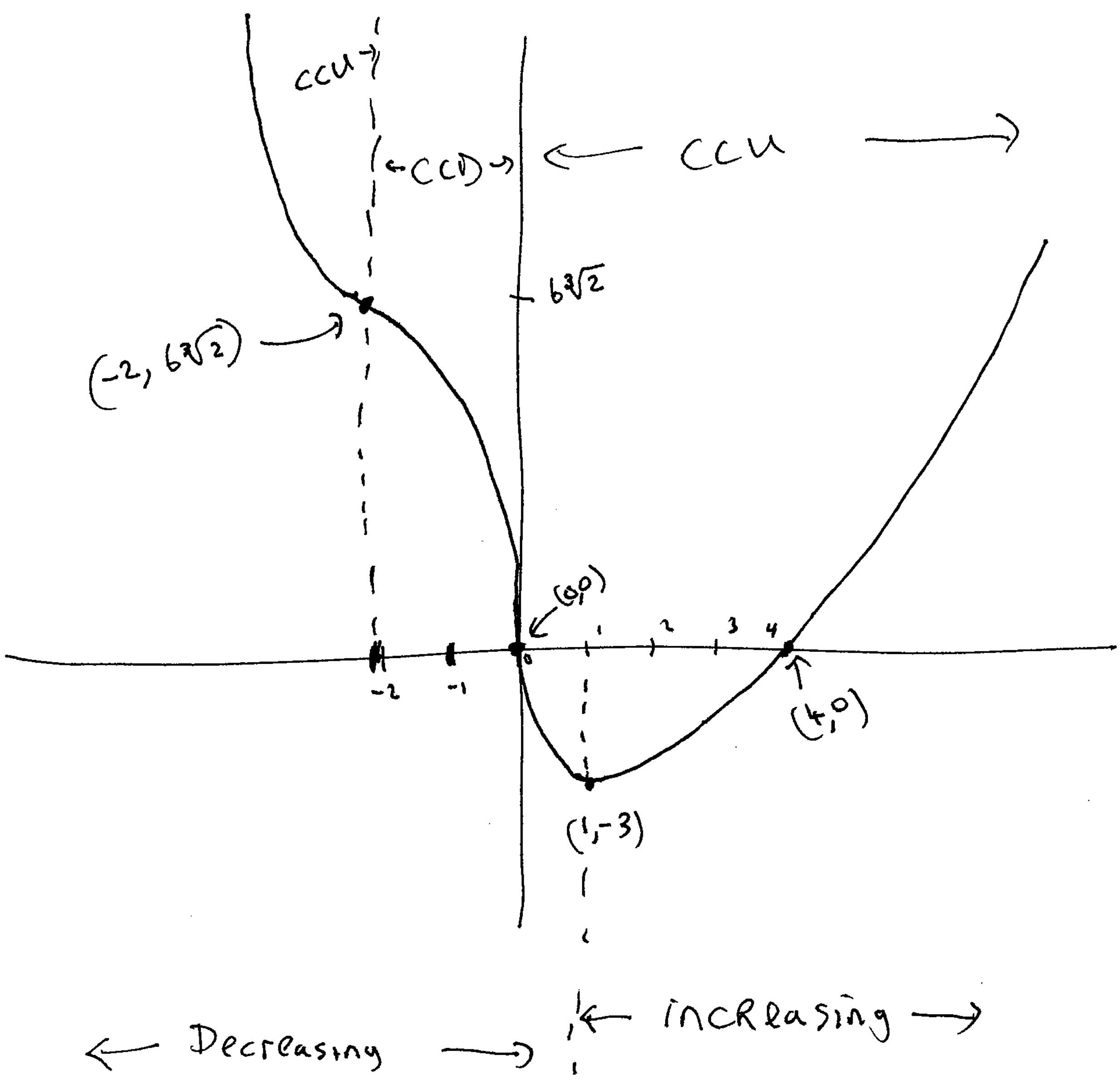
f'' DNE at $x=0$ & $f''=0$ at $x=-2$

	$(-\infty,-2)$	$(-2,0)$	$(0,\infty)$
f''	$\frac{\ominus}{\ominus} = +$	$\frac{\oplus}{\ominus} = -$	$\frac{\oplus}{\oplus} = +$
f	CCU	CCD	CCU

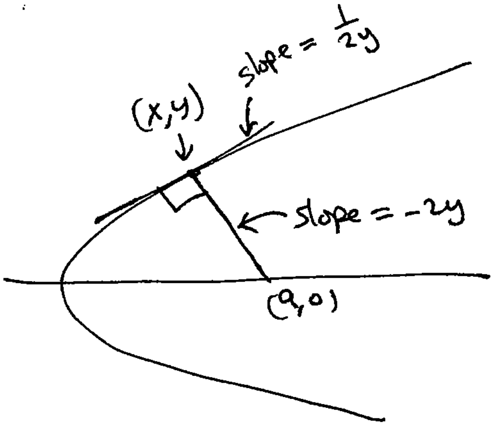
$x=0$, $x=-2$ are both inflection points.

$f(-2) = 6\sqrt[3]{2}$ $\quad f(-6) = -\frac{4}{\sqrt[3]{4}}$

Now sketch the graph $y = f(x)$:



Q3]...[12 points] Show that if it is possible to draw three normal lines from the point $(a, 0)$ to the parabola $x = y^2$, then a must be greater than 2. $\leftarrow \frac{1}{2}$ Typo!



$$x = y^2 \qquad 1 = \frac{dx}{dx} = \frac{dy^2}{dx} = 2y\,y' \leftarrow \text{implicit diff}^{n}$$

$$y' = \frac{1}{2y} \quad \text{Tangent slope}$$

$$\text{Normal slope} = -2y$$

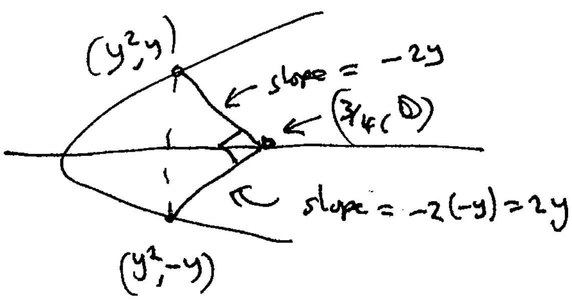
Also, $\text{Normal slope} = \dfrac{0-y}{a-x} = \dfrac{-y}{a-y^2}$

Get $\dfrac{\cancel{-y}}{a-y^2} = 2(\cancel{-y})$

$\Rightarrow 1 = 2(a-y^2) \quad \Rightarrow \boxed{a = \frac{1}{2} + y^2} > \frac{1}{2} + 0 = \frac{1}{2}$

$\boxed{\text{Need } a > \frac{1}{2}\,!}$

One of the three normals above is the x-axis. Find the value of a for which the other two normals are perpendicular to each other.



Want $(-2y)(2y) = -1 \leftarrow \perp$ lines

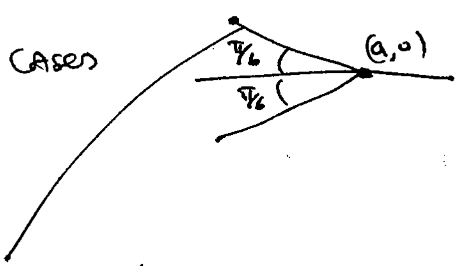
$-4y^2 = -1$

$y^2 = \frac{1}{4} \qquad y = \pm\frac{1}{2}$

$\Rightarrow a = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Find the value(s) of a for with the other two normals intersect at an angle of $\pi/3$.

Two cases



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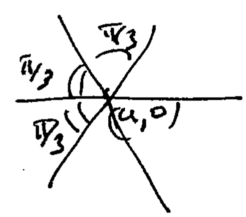
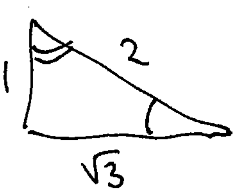


$\text{Slope} = \dfrac{-1}{\sqrt{3}} = -2y$



$y = \dfrac{1}{2\sqrt{3}}$

$y^2 = \dfrac{1}{12}$

$a = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$

$\text{Slope} = -\sqrt{3} = -2y$

$y = \dfrac{\sqrt{3}}{2}$

$a = \frac{1}{2} + y^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$

Q4]... [**8 points**] Find the absolute maximum and the absolute minimum of the function

$$f(x) = x + \frac{4}{x}$$

on the interval $[1, 6]$.

① Endpts. $x=1$ $x=6$

② Critical Pts. $f'(x) = 1 - \frac{4}{x^2}$ ... always exists on $[1,6]$

$f'(x) = 0 \Rightarrow 1 = \frac{4}{x^2}$ $4 = x^2$

$x = \pm 2$

$x = 2$ only one of ± 2 inside $[1,6]$.

③ Outputs.

$f(1) = 1 + \frac{4}{1} = 5$

$f(2) = 2 + \frac{4}{2} = 2 + 2 = 4$

$f(6) = 6 + \frac{4}{6} = 6\frac{2}{3}$

Absolute Max.

Absolute Min.