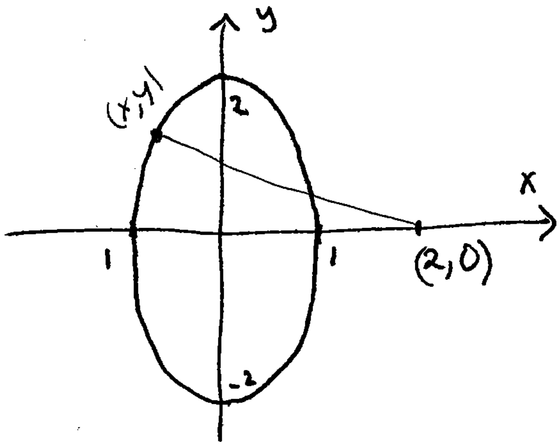
UNIV 1000 - 010 MID III

Q1]...[13 points] Find the coordinates of the points on the ellipse

$$x^2 + \frac{y^2}{4} = 1$$

which are farthest from the point $(2, 0)$.



$$(\text{dist})^2 = (x-2)^2 + y^2$$

$$f(x) = (x-2)^2 + 4 - 4x^2$$

$$f'(x) = 2(x-2) - 8x$$

$$= -4 - 6x$$

$$f'(x) = 0 \qquad x = \frac{-2}{3}$$

$$y = \frac{\pm 2\sqrt{5}}{3}$$

$$\left(\frac{-2}{3}, \pm\frac{2\sqrt{5}}{3}\right)$$

Q2]. . . **[12 points]** Express the following limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \left(\frac{2i}{n}\right)^2 \cos\left(\left(\frac{2i}{n}\right)^3\right)$$

$\Delta x \leftrightarrow \frac{2-0}{n}$ $\quad x^2 \quad x^3$

$\frac{2i}{n}$: $i=0 \Rightarrow 0$, $i=n \Rightarrow 2$

$[0,2]$ interval

$$\int_0^2 x^2 \cos(x^3)\, dx$$

Using whatever method you like, evaluate the definite integral that you obtained above.

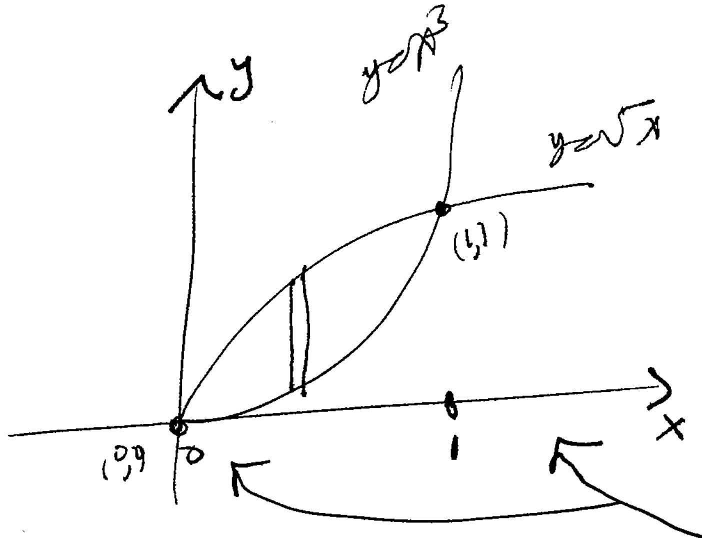
Subst$^{\underline{n}}$: $\quad u = x^3 \quad \left(\begin{array}{l} x=0 \Rightarrow u = 0^3 = 0 \\ x=2 \Rightarrow u = 2^3 = 8 \end{array} \right.$

$$du = 3x^2 dx$$

$$\int = \int_0^8 \cos(u) \frac{du}{3} = \left.\frac{\sin(u)}{3}\right]_0^3$$

$$= \frac{\sin(8)}{3}$$

Q3]... **[12 points]** Compute the area of the region between the graph of $y = \sqrt{x}$ and the graph of $y = x^3$.



Intersection of 2 graphs:

$x^3 = \sqrt{x}$

$x^6 = x$

$(x^5 - 1)x = 0$

$x = 0, \quad x^5 = 1$

$\boxed{x = 0, \quad x = 1}$

$$A = \int_0^1 x^{\frac{1}{2}} - x^3 \, dx$$

$$= \left[\frac{2}{3} x^{3/2} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$$

Q4]... **[13 points]** Write down a general expression for computing the volume of a 3-dimensional shape by slicing perpendicular to an axis.

Area of cross sectional slice $\perp$ to axis at x.

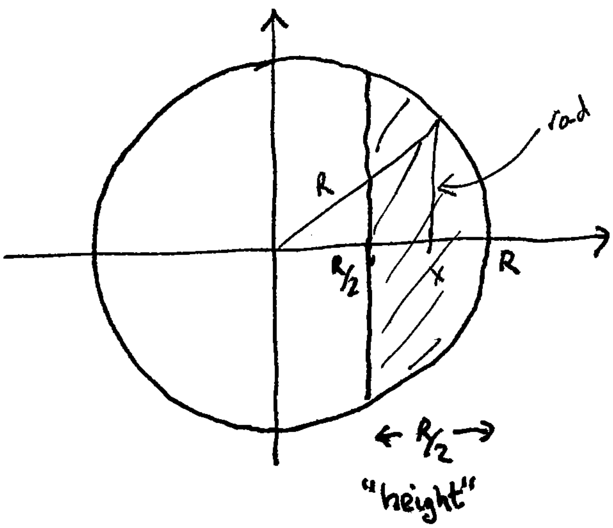
$$V = \int_a^b A(x)\,dx$$

length along the axis

$[a,b]$ = projection of object onto axis.

Compute the volume of a spherical cap of height $R/2$ and sphere radius R.

Cross - Section :



$$x^2 + (\text{rad})^2 = R^2$$

$$(\text{rad})^2 = R^2 - x^2$$

$$V = \int_{R/2}^{R} \pi\left(R^2 - x^2\right) dx$$

$$= \pi\left[R^2 x - \frac{x^3}{3}\right]_{R/2}^{R}$$

$$= \pi\left(\frac{2R^3}{3} - \frac{R^3}{2} + \frac{R^3}{24}\right)$$

$$= \frac{\pi R^3}{24}\left(8(2) - 12 + 1\right)$$

$$= \frac{5\pi R^3}{24}$$