[Mid II]: Fall 2010 – CALC I – SOLUTIONS

Q1]... [**42 points**] Compute the derivatives y' of the following.

$$y = \left(1 - \frac{x}{3}\right)^{99}$$

Chain + Power rules

$$\boxed{u = 1 - \frac{x}{3} \qquad \frac{du}{dx} = -\frac{1}{3}} \qquad \boxed{y = u^{99} \qquad \frac{dy}{du} = 99u^{98}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 99u^{98}\cdot\left(-\frac{1}{3}\right) = -33\left(1-\frac{x}{3}\right)^{98}$$

$$y = x^2 \tan(x^3)$$

PRODUCT RULE

$$\frac{dy}{dx} = \frac{dx^2}{dx}\tan(x^3) + x^2 \frac{d}{dx}\left(\tan(x^3)\right)$$

← Chain Rule &

$$\boxed{\frac{d}{d\theta}\tan\theta = \sec^2\theta}$$

Trig functions

$$= 2x\tan(x^3) + x^2 \cdot \sec^2(x^3)\frac{dx^3}{dx}$$

$$= 2x\tan(x^3) + 3x^4\sec^2(x^3)$$

$$y = \frac{\sin(3x+1)}{\sqrt{x+1}}$$

QUOTIENT RULE

$$\frac{dy}{dx} = \frac{\frac{d\sin(3x+1)}{dx}\sqrt{x+1} - \sin(3x+1)\frac{d}{dx}\left(\sqrt{x+1}\right)}{\left(\sqrt{x+1}\right)^2}$$

Chain Rule

fraction power $\left(\frac{1}{2}\right)$

trig functions

$$= \frac{\cos(3x+1)\cdot 3\cdot\sqrt{x+1} - \sin(3x+1)\frac{1}{2}(x+1)^{-\frac{1}{2}}\cdot(1)}{x+1}$$

$$= \frac{3\sqrt{x+1}\cos(3x+1) - \frac{\sin(3x+1)}{2\sqrt{x+1}}}{x+1}$$

Q2]... **[12 points]** If $y = \cos(2x+5)$ find $y^{(63)}$, the 63rd derivative of y with respect to x. Show the steps of your work clearly.

$$y = \cos(2x+5)$$

$$y' = -\sin(2x+5) \cdot \frac{d(2x+5)}{dx} = -2\sin(2x+5)$$

$$y'' = -2\cos(2x+5)\frac{d}{dx}(2x+5) = -2^2\cos(2x+5)$$

$$y^{(3)} = -2^2\left(-\sin(2x+5)\frac{d}{dx}(2x+5)\right) = +2^3\sin(2x+5)$$

$$y^{(4)} = 2^3\cos(2x+5)\frac{d}{dx}(2x+5) = 2^4\cos(2x+5)$$

2 Patterns :

1. trig portion cycles around with correct sign every 4th time.
2. Chain Rule gives $\frac{d(2x+5)}{dx} = 2$ every time

$\Rightarrow$ $y^{(n)}$ will have 2^n.

$$\boxed{y^{(63)} = 2^{63}\sin(2x+5)}$$

$63 = \cancel{64} - 1$ $\Rightarrow$ trig portion is similar to that of $y^{(3)}$.

Q3]...[22 points] The following equation defines y implicitly in terms of x.

$$xy = x + y$$

Compute the first and second derivatives y' and y'' of y with respect to x. Your answers should be expressions involving x and y only. Show the steps of your work clearly.

implicit diff ---- $\frac{d}{dx}(xy) = \frac{d}{dx}(x+y)$

product Rule   sum Rule

$$\frac{dx}{dx}y + x\frac{dy}{dx} = \frac{dx}{dx} + y'$$

$$y + xy' = 1 + y'$$

$$(x-1)y' = 1 - y \qquad \boxed{y' = \frac{1-y}{x-1}} \quad \text{—①}$$

Now $y'' = \frac{d}{dx}y' = \frac{d}{dx}\left(\frac{1-y}{x-1}\right)$ .... from ①

quotient rule

$$= \frac{\frac{d}{dx}(1-y)(x-1) - (1-y)\frac{d}{dx}(x-1)}{(x-1)^2}$$

$$= \frac{-y'(x-1) - (1-y)\,1}{(x-1)^2}$$

Substitute for y' (use ①)

$$= \frac{-\left(\frac{1-y}{\cancel{x-1}}\right)\cancel{(x-1)} - (1-y)}{(x-1)^2} = \boxed{\frac{2(y-1)}{(x-1)^2}}$$

Q4]. . . **[24 points]** Compute the following limits.

$$\lim_{x\to 2}\left(\frac{x^{99}-2^{99}}{x-2}\right)$$

= difference quotient limit!

$$= \frac{d}{dx}x^{99}\Big|_{x=2}$$

$$= 99\,x^{99-1}\Big|_{x=2}$$

$$= \boxed{99\cdot(2)^{98}}$$

$$\lim_{x\to \pi/4}\left(\frac{\cos(2x)}{x-\pi/4}\right) \longrightarrow \cos(2(\pi/4)) = \cos(\pi/2) = 0$$

$$\boxed{-2}$$

$$= \lim_{x\to\pi/4}\left(\frac{\cos(2x) - \cos(2(\pi/4))}{x - \pi/4}\right)$$

= difference quotient Limit!

$$= \frac{d}{dx}\cos(2x)\Big|_{x=\pi/4}$$

Trig + Ch. Rule

$$= -\sin(2x)\cdot\frac{d2x}{dx}\Big|_{x=\pi/4} = -2\sin(2(\pi/4)) = \boxed{-2}$$