Topology I [5853-001] Homework

A "topological proof" that there are infinitely many prime numbers.

For $n \in \mathbb{Z} \setminus \{0\}$ and $r \in \{0, \ldots, n-1\} = \{0\} \cup S_n$ define

$$B_{n,r} \;=\; \{mn + r \mid m \in \mathbb{Z}\}$$

that is, the set of integers which have remainder r on division by n. Note that $B_{1,0} = \mathbb{Z}$ while $B_{2,0}$ and $B_{2,1}$ denote the sets of even and odd integers respectively.

Complete the following steps.

Step 1. Prove that the collection

$$\mathcal{B} = \{B_{n,r} \mid n \in \mathbb{Z} \setminus \{0\}\,,\; r \in \{0\} \cup S_n\,\}$$

is a basis for a topology, $\mathcal{T}$, on $\mathbb{Z}$.

Step 2. Prove that

$$B_{n,0} \;=\; \mathbb{Z} \setminus \bigcup_{r \in S_n} B_{n,r}$$

Claim 3. Each $B_{n,0}$ is closed in this topology on $\mathbb{Z}$.

Claim 4. If there were only finitely many primes, then $\mathbb{Z} \setminus \{\pm 1\}$ is closed. [Hint: If the list of all primes is $p_1, \ldots, p_k$, then then show that $\mathbb{Z} \setminus \{\pm 1\}$ is equal to $B_{p_1,0} \cup \cdots \cup B_{p_k,0}$]

Claim 5. Conclude that $\{\pm 1\}$ is open, and that this is a contradiction (why?).