

Applications of S-B (Schröder-Bernstein Theorem)

①

Prop $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.

App ①

PF Define $f: \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$
 $: n \mapsto (n, 1)$

$f(n) = f(m) \Rightarrow (n, 1) = (m, 1)$
 $\Rightarrow n = m$... defⁿ of equality of ordered pairs.

$\Rightarrow f$ injective.

Define $g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$
 $: (n, m) \mapsto 3^n 5^m$

$g(n, m) = g(a, b) \Rightarrow 3^n 5^m = 3^a 5^b$

$\Rightarrow n = a, m = b$... by uniqueness in F.T.A.

$\Rightarrow (n, m) = (a, b)$ -- by defⁿ of equality of ordered pairs.

$\Rightarrow g$ injective.

S-B theorem $\Rightarrow \exists$ bijection $\mathbb{N} \approx \mathbb{N} \times \mathbb{N}$.

□

Th^m

$$P(\mathbb{N}) \approx \mathbb{R}$$

App ②

①

Proof

We have seen before that

$$P(\mathbb{N}) \approx S = \{ \text{infinite binary strings} \} \\ = \{ a_1 a_2 a_3 \dots \mid a_i \in \{0,1\}, \forall i \in \mathbb{N} \}$$

By the Schröder - Bernstein Th^m it suffices to produce two injective functions:

$$f: S \longrightarrow \mathbb{R}$$

$$g: \mathbb{R} \longrightarrow S$$

f

$$f: S \longrightarrow \mathbb{R}$$

$$: a_1 a_2 a_3 \dots \longmapsto 0.5 a_1 a_2 a_3 \dots$$

$$f(a_1 a_2 \dots) = g(b_1 b_2 \dots)$$

$$\Rightarrow 0.5 a_1 a_2 \dots = 0.5 b_1 b_2 \dots$$

$$\Rightarrow a_1 = b_1, a_2 = b_2, \dots, a_i = b_i \forall i \in \mathbb{N}.$$

This is because the only ambiguity with decimal expansions of real numbers is when one of the decimals ends with an ∞ string of 9's. This is not the case here.

$$\Rightarrow a_1 a_2 \dots = b_1 b_2 \dots$$

$\Rightarrow f$ injective!

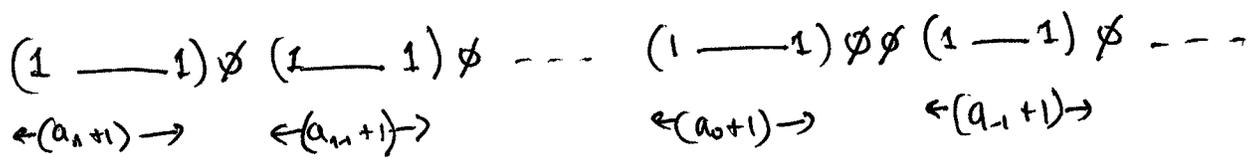
9) We "encode" decimal expansions of real numbers using binary strings as follows.

- $a_j = \text{digit} \longleftrightarrow \text{encode by } (a_{j+1}) \text{ 1's}$
- $\text{place between digits} \longleftrightarrow \text{encode by } \emptyset$
- $\text{decimal point} \longleftrightarrow \text{encode by } \emptyset\emptyset$.

Fact: Each real number has a unique ∞ decimal number representation (possibly ending in ∞ string of \emptyset 's), which does not end in ∞ seq. of 9 's.

- eg
- $3.14159 \dots \longleftrightarrow 1111 \emptyset\emptyset 11 \emptyset 11111 \emptyset 11 \emptyset 111111 \emptyset 1 \dots$
 - $32 = 32.000\dots \longleftrightarrow 1111 \emptyset 111 \emptyset\emptyset 1 \emptyset 1 \emptyset 1 \emptyset \dots$
 - $0.306 \longleftrightarrow 1 \emptyset\emptyset 1111 \emptyset 1 \emptyset 1111111 \emptyset 1 \emptyset 1 \emptyset 1 \emptyset \dots$
 - $0.36 \longleftrightarrow 1 \emptyset\emptyset 1111 \emptyset 1111111 \emptyset 1 \emptyset 1 \emptyset 1 \emptyset \dots$

$$a_n \dots a_0 \cdot a_{-1} a_{-2} \dots \longleftrightarrow$$



$g: \mathbb{R} \rightarrow S$
 $\vdots x \longmapsto \text{encoded version of } x.$

By construction/definition of the encoding

(3)

$g(x) = g(y) \Rightarrow x$ & y have exactly same digits before + after the decimal point.

$$\Rightarrow x = y$$

$\Rightarrow g$ injective.

S-B theorem $\Rightarrow S \approx \mathbb{R}$

□

Proof $(0,1) \approx (0,1) \times (0,1)$

App. ③

where $(0,1)$ is the interval $0 < x < 1$ on the real line.

Proof $f: (0,1) \longrightarrow (0,1) \times (0,1)$
 $: x \longmapsto (x, \frac{1}{2})$

$$f(x) = f(y) \Rightarrow (x, \frac{1}{2}) = (y, \frac{1}{2})$$

$\Rightarrow x = y$ --- defⁿ of equality of ordered pairs

$\Rightarrow f$ injective.

