The set $\mathbb{Z}$ is equivalent to the set E of even integers. ①

Proof Consider the function $f : \mathbb{Z} \longrightarrow E$
$: n \longmapsto 2n$

① Claim : f is injective.

$$f(n) = f(m) \Rightarrow 2n = 2m \Rightarrow \frac{2n}{2} = \frac{2m}{2} \Rightarrow n = m.$$

② Claim : f is surjective.

By def^n of even integer $m \in E$ means $m = 2k$ for some $k \in \mathbb{Z}$.

But then $m = f(k)$.

①&② $\Rightarrow f$ bijective $\Rightarrow \mathbb{Z} \approx E$.

The set $\mathbb{Z}$ is equivalent to the set T of integers which are divisible by 3.

$$T = \{3m \mid m \in \mathbb{Z}\}$$

Proof Consider the function $g : \mathbb{Z} \longrightarrow T : n \longmapsto 3n$

① g is injective $g(m) = g(n) \Rightarrow 3m = 3n \Rightarrow \frac{3m}{3} = \frac{3n}{3}$
$\Rightarrow m = n$.

② g is surjective By definition of T, $m \in T$ means $m = 3k$ for some $k \in \mathbb{Z}$.

But this means $m = g(k)$.

①&② $\Rightarrow g$ bijective $\Rightarrow \mathbb{Z} \approx T$.

The interval $(0,1)$ is equivalent to the interval (a,b) for any $a<b\in\mathbb{R}$.

Proof

length of $(0,1) = 1-0 = 1$ length of $(a,b) = b-a$

So we must stretch by a factor of $(b-a)$; i.e. $x \mapsto (b-a)x$

Pure stretching takes $(0,1)$ to $(0, b-a)$ and we must add a to move this interval to $(0+a, b-a+a) = (a,b)$.

This reasoning suggests using the following function

$$\begin{aligned} f &: (0,1) \longrightarrow (a,b) \\ &: x \longmapsto (b-a)x + a \end{aligned}$$

① & ② below $\Rightarrow f$ bijective. $\Rightarrow (0,1) \approx (a,b)$

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① f injective

$$f(x) = f(y) \Rightarrow (b-a)x+a = (b-a)y+a$$

$$\Rightarrow (b-a)x + a - a = (b-a)y + a - a$$

$$\Rightarrow (b-a)x = (b-a)y$$

It is legal to divide by $b-a$ since we were told $a<b$... $\Rightarrow \boxed{b-a>0}$

$$\Rightarrow \frac{(b-a)x}{(b-a)} = \frac{(b-a)y}{(b-a)}$$

$$\Rightarrow x = y$$

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② f surjective given $y\in(a,b) \Rightarrow a<y<b$

$$\Rightarrow 0 < y-a < b-a$$

$$\Rightarrow 0 < \frac{y-a}{b-a} < 1$$

Call this x.

$x\in(0,1)$ and $f(x) = (b-a)x+a = \cancel{(b-a)}\left(\frac{y-a}{\cancel{b-a}}\right)+a = y-a+a = y.$

$\boxtimes$

The interval $(0,1)$ is equivalent to $\mathbb{R}$.

Pf ① By a previous result we know $(0,1) \approx (-\pi/2, \pi/2)$

$$f : (0,1) \longrightarrow (-\pi/2, \pi/2) \ : \ x \longmapsto \pi x - \pi/2$$

② The function $g : x \longmapsto \tan(x)$ takes $(-\pi/2, \pi/2)$ to $\mathbb{R}$.

g injective. $\frac{dg}{dx} = \frac{d\tan(x)}{dx} = \sec^2(x) > 0$ on $(-\pi/2, \pi/2)$.

Calc. (M.V.T.) $\Rightarrow$ $g(x)$ increasing on $(-\pi/2, \pi/2)$

$\Rightarrow$ $g(x)$ injective on $(-\pi/2, \pi/2)$.

g surjective.

- $\lim_{x \to \pi/2^-} g(x) = +\infty$ — First Fact
- $\lim_{x \to -\pi/2^+} g(x) = -\infty$ — Second Fact
- Intermediate value Theorem ($g(x)$ continuous) — Third Fact

Use these 3 results from calc class to argue that $g(x)$ is surjective.

given $y \in \mathbb{R}$ 1st fact $\Rightarrow \exists\, b \in (-\pi/2, \pi/2)$ so that $g(b) > y$

2nd fact $\Rightarrow \exists\, a \in (-\pi/2, \pi/2)$ so that $g(a) < y$.

Now $a < b$ (because $g(x)$ increasing) and I.V.T. applied to $g(x)$ on $[a,b]$ $\Rightarrow \exists\, c$ in (a,b) so that

$$g(c) = y.$$

$g(a) < y$
$g(b) > y$

Now g inj. & g surj. $\Rightarrow$ g bijective

$\Rightarrow (-\pi/2, \pi/2) \approx \mathbb{R}$.

Now combine 1 & 2 (composition of bijections gives a bijection) to get $(0,1) \approx \mathbb{R}$

$$(0,1) \xrightarrow{\ f\ } (-\pi/2, \pi/2) \xrightarrow{\ g\ } \mathbb{R}$$

$$x \longmapsto \pi x - \pi/2$$

$$x \longmapsto \tan(x)$$

Here is the composed map

$$\boxed{\begin{array}{ccc} (0,1) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \tan\left(\pi x - \pi/2\right) \end{array}}$$

explicit bijection between $(0,1)$ and $\mathbb{R}$.

If A, B are both countably ∞ and $A \cap B = \emptyset$, then $A \cup B$ is countably ∞.

Proof By def^{n} $\exists$ bijection $f: A \to \mathbb{N}$ and $\exists$ bijection $g: B \to \mathbb{N}$.

Now define $h: A \cup B \to \mathbb{N}$ by

$$h(x) = \begin{cases} 2f(x) & \text{if } x \in A \\ 2g(x) - 1 & \text{if } x \in B \end{cases}$$

This is a good, piecewise-defined function since $A \cap B = \emptyset$.

h is injective Assume $h(x) = h(y)$. There are 2 cases to consider,

$h(x) = h(y)$ is even	$h(x) = h(y)$ is odd.
$\Rightarrow 2f(x) = 2f(y)$	$\Rightarrow 2g(x) - 1 = 2g(y) - 1$
$\Rightarrow f(x) = f(y)$	$\Rightarrow 2g(x) = 2g(y)$
	$\Rightarrow g(x) = g(y)$
$\Rightarrow x = y$ --- since f is injective	$\Rightarrow x = y$ --- since g is injective.

In each case, we conclude that $x = y$.
Thus $h(x) = h(y) \Longrightarrow x = y$, & so h is injective.

h is surjective

Given $n \in \mathbb{N}$, there are 2 cases

- n is even $\Rightarrow$ $n = 2M$ for some $M \in \mathbb{N}$.
  $f(x)$ surjective $\Rightarrow$ $\exists\, x \in A \subseteq A \cup B$ so that $f(x) = M$
  $\Rightarrow h(x) = 2f(x) = 2M = n$.

- n is odd $\Rightarrow$ $n = 2q - 1$ for some $q \in \mathbb{N}$
  g surjective $\Rightarrow$ $q = g(x)$ for some $x \in B \subseteq A \cup B$
  $\Rightarrow h(x) = 2g(x) - 1 = 2q - 1 = n$.

In each case, n is the image of some element in $A \cup B$ under h. $\Rightarrow$ h is surjective.

$\mathbb{Z}$ is equivalent to $\mathbb{N}$.

Proof Consider the following piecewise defined function:

$$f: \mathbb{Z} \longrightarrow \mathbb{N} : n \longmapsto \begin{cases} 2n+1 & (n \geq 0) \\ -2n & (n < 0) \end{cases}$$

Claim f is injective. Suppose $f(a) = f(b)$. There are 2 cases

CASE ① $f(a) = f(b)$ is odd

$\Rightarrow 2a+1 = 2b+1$

$\Rightarrow (2a+1)-1 = (2b+1)-1$

$\Rightarrow 2a = 2b$

$\Rightarrow \frac{2a}{2} = \frac{2b}{2}$

$\Rightarrow a = b$

CASE ② $f(a) = f(b)$ is even.

$\Rightarrow -2a = -2b$

$\Rightarrow \frac{-2a}{-2} = \frac{-2b}{-2}$

$\Rightarrow a = b$

In each case we obtain the conclusion : $a = b$.

Thus $(f(a) = f(b)) \rightarrow (a = b)$, and so f is injective.

Claim f is surjective

Given $m \in \mathbb{N}$ there are 2 cases

CASE ① m is even $\Rightarrow m = 2p$ for some $p \in \mathbb{N}$

$\Rightarrow m = f(-p)$.

CASE ② m is odd $\Rightarrow m = 2q+1$ for some $q \in \mathbb{Z}$, $q \geq 0$

$\Rightarrow m = f(q)$.

$\Rightarrow$ In each case m is an output of f.

$\Rightarrow$ f is onto.

QED