

Goal : Make 4 gal. from 3 gal. & 5 gal. containers

Observation: amount in each container at any stage is an integer linear combination of 3 & 5.

$$(2)(3) + (-1)(5) = 6 - 5 = 1 \Rightarrow \text{we can make 1 gal.}$$

$$1 + 3 = 4 \Rightarrow \text{we can make 4 gal.}$$

↙ here's one way!

| Step # | 3 gal. | 5 gal. |
|--------|------------------|---|
| Start | 0 | 0 |
| 1 | 3 | 0 |
| 2 | 0 | 3 |
| 3 | 3 | 3 |
| 4 | $1 = (2)(3) - 5$ | 5 |
| 5 | 1 | 0 |
| 6 | 0 | 1 |
| 7 | 3 | 1 |
| 8 | 0 | $\textcircled{4} = 1 + 3 = (2)(3) + (-1)(5) + 3 = (3)(3) + (-1)(5)$ |

Bruce + Samuel save the city!

Goal Make 8 gal. from 6 gal. and 21 gal containers.

Observation amount in each container at any stage is an integral linear combination, $l(6) + m(21)$, of 6 & 21.

Note $3|6$ & $3|21$

$\Rightarrow 3|l(6) + m(21)$ for any $m, n \in \mathbb{Z}$

Now $3 \nmid 8$

Therefore it's impossible to express 8 as an integral linear combination of 6 and 21.

\Rightarrow Can't obtain 8 gal. from 6 gal & 21 gal containers!

Bruce + Samuel not so lucky ---!