

A proof that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$, $\forall n \in \mathbb{N}$

using the Least Principle.

call this $P(n)$

Proof Let $T = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$

and $F = \{n \in \mathbb{N} \mid P(n) \text{ is false}\}$

Clearly, given any $n \in \mathbb{N}$ it either belongs to T or to F (but not both).

We want to prove that F is empty (and so T is all of \mathbb{N}).

Argue by contradiction. Assume F is nonempty.

By the Least Principle, F has a smallest element — call it n_0 .

Claim ① $n_0 \neq 1$

This is because $1 \in T$

because $1 = \frac{1(1+1)}{2}$

$1 = 1$ yes!

Therefore $n_0 \geq 2$

and we can subtract 1 & say inside of \mathbb{N} .

$$n_0 - 1 \geq 1 \quad \text{so } n_0 - 1 \in \mathbb{N}.$$

But $n_0 - 1 \notin F$ since n_0 was least element of F .

$\Rightarrow P(n_0 - 1)$ holds true.

$$\text{i.e. } 1 + 2 + \dots + (n_0 - 1) = \frac{(n_0 - 1)((n_0 - 1) + 1)}{2}$$

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$$\text{i.e. } 1 + 2 + \dots + (n_0 - 1) = \frac{(n_0 - 1) n_0}{2}$$

Add n_0 to both sides.

$$\begin{aligned} \Rightarrow 1 + 2 + \dots + (n_0 - 1) + n_0 &= \frac{(n_0 - 1) n_0}{2} + n_0 \\ &= \frac{(n_0 - 1) n_0 + 2 n_0}{2} \\ &= \frac{((n_0 - 1) + 2) n_0}{2} \end{aligned}$$

$$\text{i.e. } 1 + 2 + \dots + n_0 = \frac{(n_0 + 1) n_0}{2}$$

$$= \frac{n_0 (n_0 + 1)}{2}$$

i.e. $P(n_0)$ is true

$\Rightarrow n_0 \notin F$ a contradiction!

This contradiction arose from the assumption that F is nonempty.

Therefore F is empty, and $T = \mathbb{N}$.

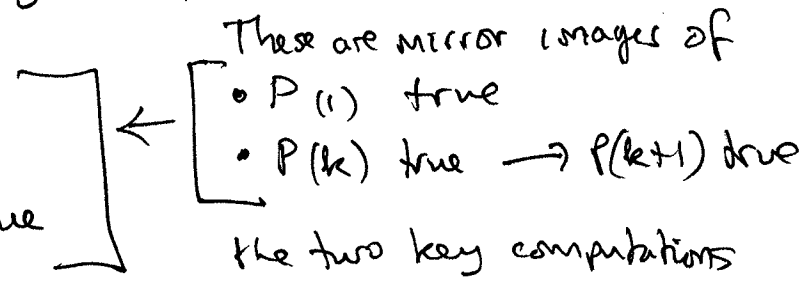
i.e. $P(n)$ is true, $\forall n \in \mathbb{N}$.



Remark / Framework seems completely different to induction \rightarrow we're using a "proof by contradiction" technique.

However the two key computations we made were:

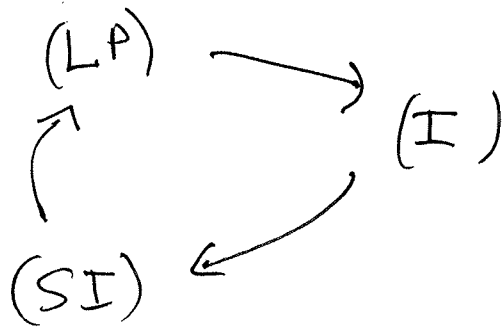
- $P(1)$ true
- $P(n-1)$ true $\rightarrow P(n)$ true



the two key computations that you do in a proof by induction.

So it should not come as a surprise that the Least Principle (LP) implies the principle of Mathematical induction (I). You may even have an idea how to prove (I) using (LP)

In the Least Principle handout we show that the following implications hold:



Here (SI) denotes Strong Induction. Thus all three principles are saying the same thing about the set, \mathbb{N} , of positive integers.

