①

A proof that $\underbrace{1+2+\cdots+n = \frac{n(n+1)}{2}}_{\text{call this } P(n)}$, $\forall n \in \mathbb{N}$

using the <u>Least Principle</u>.

<u>Proof</u> Let $T = \{ n \in \mathbb{N} \mid P(n) \text{ is true} \}$

and $F = \{ n \in \mathbb{N} \mid P(n) \text{ is false} \}$

Clearly, given any $n \in \mathbb{N}$ it either belongs to T or to F (but not both).

We want to prove that F is empty (and so T is all of $\mathbb{N}$).

Argue by contradiction. Assume F is nonempty.

By the Least Principle, F has a smallest element — call it n_0.

<u>Claim ①</u> $n_0 \neq 1$

This is because $1 \in T$

because $1 = \frac{1(1+1)}{2}$

$1 = 1$ yes!

Therefore $n_0 \geqslant 2$

and we can subtract 1 & say inside of $\mathbb{N}$.

$n_0 - 1 \geqslant 1$ so $n_0 - 1 \in \mathbb{N}$.

But $n_0 - 1 \notin F$ since n_0 was least element of F.

$\Rightarrow P(n_0 - 1)$ holds true.

i.e. $1 + 2 + \cdots + (n_0 - 1) = \dfrac{(n_0-1)((n_0-1)+1)}{2}$

i.e. $1 + 2 + \cdots + (n_0 - 1) = \dfrac{(n_0-1)\, n_0}{2}$

Add n_0 to both sides so to

$$\Rightarrow 1 + 2 + \cdots + (n_0 - 1) + n_0 = \frac{(n_0-1)\,n_0}{2} + n_0$$

$$= \frac{(n_0-1)\,n_0 + 2n_0}{2}$$

$$= \frac{((n_0-1)+2)\,n_0}{2}$$

i.e. $1 + 2 + \cdots + n_0 = \frac{(n_0+1)\, n_0}{2}$

$= \frac{n_0\,(n_0+1)}{2}$

i.e. $P(n_0)$ is true

$\Rightarrow n_0 \notin F$ a contradiction!

This contradiction arose from the assumption that F is nonempty.

Therefore F is empty, and $T = \mathbb{N}$.

i.e. $P(n)$ is true, $\forall n \in \mathbb{N}$.

$\boxed{}$

<u>Remark</u>/. Framework seems completely different to induction $\rightarrow$ we're using a "proof by contradiction" technique.

However the two key computations we made were:

- $P(1)$ true
- $P(n_0-1)$ true $\rightarrow$ $P(n_0)$ true

These are mirror images of

- $P(1)$ true
- $P(k)$ true $\rightarrow$ $P(k+1)$ true

the two key computations that you do in a proof by induction.

So it should not come as a surprise that the Least Principle (LP) implies the principle of Mathematical induction (I). You may even have an idea how to prove (I) using (LP) - - - -

In the Least Principle handout we show that the following implications hold:

$$\begin{tikzcd}
(LP) \arrow[dr] & \\
& (I) \arrow[dl] \\
(SI) \arrow[uu] &
\end{tikzcd}$$

Here (SI) denotes Strong Induction. Thus all three principles are saying the same thing about the set, $\mathbb{N}$, of positive integers.