2513 Mid I — Solutions

Q1]... [25 points]

1. Give the definition of an *odd integer.*

An integer n is said to be odd if
$n = 2q+1$ for some other integer q.

2. Give a detailed proof of the following proposition about integers n.

If n is odd, then n^2 is odd.

We are given that n is odd. This means
$n = 2q+1$ for some integer q.

Squaring gives
$$n^2 = (2q+1)^2 = 4q^2+4q+1 = 2(2q^2+2q)+1.$$

Note that $k = 2q^2+2q$ is an integer, since $\mathbb{Z}$ is closed under mult^n and under add^n.

Thus we have expressed n^2 as $2k+1$ for some integer k, and conclude that n^2 is odd. $\boxtimes$

3. Is the following proposition about integers n true or false? Why?

If n^2 is even, then n is even.

it is TRUE.

It is logically equivalent to its contrapositive; namely
"If n is not even, then n^2 is not even".

In other words
"If n is odd, then n^2 is odd,"
which is true & was proven in part 2 above.

Q2]... [25 points]

1. Write down the *converse* of the conditional statement $P \longrightarrow Q$.

$$Q \rightarrow P$$

2. Write down the *contrapositive* of the conditional statement $P \longrightarrow Q$.

$$(\neg Q) \rightarrow (\neg P)$$

3. Which of the two statements above are logically equivalent to the original statement $P \longrightarrow Q$?

$(\neg Q) \rightarrow (\neg P)$ is.

4. For each of the following statements, say whether it is equivalent to the negation of a conditional: $\neg(P \longrightarrow Q)$. Give reasons for your answers.

P	Q	$\neg P$	$\neg Q$	(a)	(b)	(c)	(d)	$P\to Q$	$\neg(P\to Q)$
T	T	F	F	T	F	F	T	T	F
T	F	F	T	F	F	T	T	F	T
F	T	T	F	T	T	F	F	T	F
F	F	T	T	T	F	F	T	T	F

only (c) agrees!

Columns (a),(b),(d) do not agree with final column.

Reason: Truth tables are the safest argument!

(a) $\neg P \vee Q$

No

(b) $\neg P \wedge Q$

No

(c) $P \wedge \neg Q$

Yes

(d) $P \vee \neg Q$

NO

Q3]...[25 points] Give a careful proof of the following proposition about real numbers x and y. If it helps, you may use the fact that the product of an arbitrary real number and 0 is equal to 0.

If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

This is equivalen to the (contrapositive) statement:

"If $xy = 0$, then $x = 0$ OR $y = 0$."

This latter statement is equivalent to the statement:

"If $xy = 0$ and $x \neq 0$, then $y = 0$."

(by $(P \to Q \vee R) \equiv (P \wedge \neg Q) \to R$)

Assume $xy = 0$ and $x \neq 0$ (hypotheses).

$x \neq 0 \Rightarrow$ we can work with $\frac{1}{x}$ (from table 1.2... Mult. inverses)

$\frac{1}{x}(x \cdot y) = \frac{1}{x} \cdot (0)$

$\left(\frac{1}{x} \cdot x\right) \cdot y = \left(\frac{1}{x}\right) \cdot (0) = 0$ ... we are told we can ~~assume~~ use the fact (any real #) $\cdot (0) = 0$

$\Rightarrow 1 \cdot y = 0$

$\Rightarrow y = 0$

$\boxtimes$

If you didn't recall that

$$P \to (Q \vee R) \equiv (P \wedge \neg Q) \to R$$

all is not lost.

Want to show...

"If $xy=0$, then $x=0$ or $y=0$."

$\cancel{\text{L}}$

<u>pf</u> There are 2 possibilities; $x=0$ or $x \neq 0$.

If $x=0$, then "$x=0 \vee y=0$" holds and we are done in this case.

If $x \neq 0$, then $\frac{1}{x}$ exists (mult. inverse property) and multiplying $xy=0$ across by $\frac{1}{x}$ gives

$$\frac{1}{x} \cdot (x \cdot y) = \frac{1}{x} \cdot 0 = 0$$

told we can use this fact: $\left(\frac{1}{x}\right)(0) = 0$

$$\Rightarrow \left(\frac{1}{x} \cdot x\right) \cdot y = 0$$

$$\Rightarrow 1 \cdot y = 0$$

$$\Rightarrow y = 0$$

$\Rightarrow$ "$x=0 \vee y=0$" holds, and we are done in this case too.

$\boxed{\text{///}}$

Q4]... **[25 points]** Let $P(x, y)$ be the predicate $x \leq y$. Say which of the following quantified statements are true for the universal set $\mathbb{N}$ of all positive integers. Give reasons to support your answers.

1. $(\forall x \in \mathbb{N})(\forall y \in \mathbb{N})P(x, y)$

FALSE Here is a counterexample: $x = 7$, $y = 3$.

2. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})P(x, y)$

TRUE We can take $y = x$.

$x = x \Rightarrow x \leq x$ for every x in $\mathbb{N}$.

3. $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})P(x, y)$

TRUE $x = 1$ is the least element of $\mathbb{N}$.

$1 \leq y$ for all $y \in \mathbb{N}$.

4. $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})P(x, y)$

TRUE One example is enough to establish the truth of this existential statement.

e.g., $x = 3$, $y = 50$.

Write down the negation of the statement $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})P(x, y)$.

$$(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})(x > y).$$

(This is FALSE).