Prop ① Suppose $S \subseteq \mathbb{N}$ is an infinite subset. Then S is countably infinite.

Proof Define $f: \mathbb{N} \to S$ as follows:

$f(1) =$ least element of S --- (by L.P.).

$f(2) =$ least element of $(S - \{f(1)\})$ ---. $S - \{f(1)\} \neq \emptyset$ because S is infinite.

$\vdots$

$f(n) =$ least element of $(S - \{f(1), \ldots, f(n-1)\})$.

f injective $\boxed{\text{by definition}}$ Suppose $m < n$ then $f(m)$ chosen before $f(n)$ &

by def= $f(n)$ is chosen from $S - \{f(1), \ldots, f(n-1)\}$

$f(m)$ in here!

$\Rightarrow f(n) \neq f(m)$.

f surjective $\boxed{\text{again by definition of } f}$

Key Fact: $m \in S$ will appear among the elements $\{f(1), f(2), \ldots, f(m)\}$

Why: Well m is the m^{th} - least element of $\mathbb{N} \Rightarrow$ will be somewhere in the range of 1^{st} to m^{th} least element of S.

∎

Prop ② A countably infinite and $B \subseteq A$ - infinite,

$\implies$ B countably infinite.

Proof By def$^{=}$ of countably infinite, $\exists$ bijection

$$g: A \longrightarrow \mathbb{N}.$$

Now $g|_B : B \longrightarrow g(B) \subseteq \mathbb{N}$

is a bijection from B to the subset $g(B)$

$= \{ g(b) \mid b \in B \}$ of $\mathbb{N}$.

B infinite $\Rightarrow$ $g(B)$ is an infinite subset of $\mathbb{N}$.

Prop ① $\Rightarrow$ $\exists$ bijection $f: \mathbb{N} \longrightarrow g(B)$.

Combining these 2 bijections

$$\mathbb{N} \xrightarrow{f} g(B) \xleftarrow{g|_B} B$$

gives a bijection $(g|_B)^{-1} \circ f$ from $\mathbb{N}$ to B.

$\Rightarrow$ B is countably infinite.

$\square$

The set S of all ∞ strings of 0's and 1's is infinite, but is not equivalent to $\mathbb{N}$.

It is said to be <u>uncountably infinite</u> (<u>uncountable</u> for short).

<u>Proof</u> ① <u>S is infinite.</u>

The following is an infinite list of elements in S, so S is clearly an infinite set.

$$\{100\cdots,\ 0100\cdots,\ 0010\cdots,\ \cdots\}$$

Strings which are all 0's except for a single 1 in the n^{th} position for each positive integer n.

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② <u>$\nexists$ bijection $\mathbb{N} \longrightarrow S$.</u>

We will argue that $\nexists$ surjection $f: \mathbb{N} \to S$ (hence $\nexists$ bijection $\mathbb{N} \to S$). "Intuitively" S has too many elements."

We argue directly as follows.

Consider <u>any</u> function $f: \mathbb{N} \to S$. We will show that f is <u>not</u> <u>surjective</u> by finding elements of S which are not in the image of f. Here goes $\longrightarrow$

Write out all the strings that are images of f in a big table as shown:

$$f(1) = \boxed{0}\,0\,1\,1\,0\,1\,0\,----$$
$$f(2) = 1\,\boxed{1}\,1\,1\,0\,0\,0\,----$$
$$f(3) = 0\,0\,\boxed{0}\,0\,0\,0\,0\,---$$
$$f(4) = 1\,1\,0\,\boxed{1}\,0\,1\,0\,1\,----$$
$$\vdots$$

Build a new element of S as follows: it's n^{th} place digit is 0 if the n^{th} place digit of $f(n)$ is 1, it's n^{th} place digit is 1 if the n^{th} place digit of $f(n)$ is 0!

Look at diagonal string

$$\boxed{0}\,\boxed{1}\,\boxed{0}\,\boxed{1}\,----$$

& swap every digit to get the new element s, of S

$$s = 1\,0\,1\,0\,-----$$

Note $s \neq f(1)$ since they don't agree on 1^{st} digit

$s \neq f(2)$ — — — — — — — 2^{nd} digit

$s \neq f(3)$ — — — — — — — 3^{rd} digit

$\vdots$

$s \neq f(n)$ — — — — — — — n^{th} place digit.

$\Rightarrow \; s \neq f(n)$ for any $n \in \mathbb{N}$.

$\Rightarrow \; s \notin \text{Image}(f)$

$\Rightarrow \; f$ is <u>NOT</u> surjective. $\boxed{//}$

<u>Corollary</u>. The power set of $\mathbb{N}$ is uncountably infinite.

<u>Proof</u>

$$\mathcal{P}(\mathbb{N}) \cong \{0,1\}^{\mathbb{N}}$$ ... seen earlier

Subset $A \subseteq \mathbb{N} \rightsquigarrow \chi_A$ characteristic function

$$\cong \{(0,1,\ldots)\}$$

infinite-tuples of 0's & 1's }

$$\cong \{\text{infinite strings of 0's \& 1's}\}$$

& we've seen that this is uncountably infinite!
