PRINT NAME: <u>SOLUTIONS</u>

Calculus IV [2443–002] Midterm I

Q1]... *Find an equation for the tangent plane to the graph of $f(x, y) = x^2 + 2xy - y^2$ at the point $(2, 1, 7)$.*

Ans: Equation is given by $(z - z_0) = f_x(2, 1)(x - x_0) + f_y(2, 1)(y - y_0)$. We have $f_x = 2x + 2y$ and $f_y = 2x - 2y$ which gives us $f_x(2, 1) = 6$ and $f_y(2, 1) = 2$.

Thus, our equation becomes $(z - 7) = 6(x - 2) + 2(y - 1)$ which simplifies to

$$z \;=\; 6x + 2y - 7\,.$$

The graph of $f(x, y) = x^2 + 2xy - y^2$ and the vertical plane $y = 1$ intersect in a curve. Find a parametric equation for the tangent line to this curve of intersection at the point $(2, 1, 7)$.

Ans: We have a point on the line; namely $(2, 1, 7)$. We only have to find a parallel vector, $\mathbf{V}$. First, note that $\mathbf{V}$ lies in the plane $y = 1$ and so its y-component will be zero (it is perpendicular to the y-axis). So we can write $\mathbf{V} = \langle a, 0, b\rangle$. Now, the ratio b/a is just the slope of the tangent vector to the curve of intersection in the $y = 1$ plane. This is just $f_x(2, 1)$ from the definition of f_x. Thus, if we let $a = 1$ then we get $c = f_x(2, 1) = 6$ from part one above. So $\mathbf{V} = \langle 1, 0, 6\rangle$.

Another way to obtain $\mathbf{V}$ is to think of our curve of intersection as a parametric curve obtained by setting $y = 1$ as shown: $\mathbf{r}(x) = \langle x, 1, f(x, 1)\rangle$. Then we can take $\mathbf{V}$ to be the tangent vector

$$\left.\frac{d\mathbf{r}}{dx}\right|_{x=2} \;=\; \langle 1, 0, f_x(2, 1)\rangle \;=\; \langle 1, 0, 6\rangle\,.$$

Finally, we write out the equation of the line as

$$\langle x, y, z\rangle \;=\; \langle 2, 1, 7\rangle + t\langle 1, 0, 6\rangle$$

or

$$x \;=\; 2 + t \qquad\qquad y \;=\; 1 \qquad\qquad z \;=\; 7 + 6t\,.$$

Q2]... *Recall that Cartesian coordinates can be viewed as functions of polar coordinates as shown:*

$$x \;=\; r\cos\theta \qquad\qquad\qquad y \;=\; r\sin\theta\,.$$

Suppose $f(x, y)$ is a differentiable function of x and y. Use the Chain Rule to express $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ in terms of f_x and f_y.

Ans: First note that $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$ and $\frac{\partial y}{\partial \theta} = r\cos\theta$.

By the chain rule we have

$$\frac{\partial f}{\partial r} \;=\; \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} \;=\; f_x\cos\theta \;+\; f_y\sin\theta$$

and

$$\frac{\partial f}{\partial \theta} \;=\; \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} \;=\; -f_x r\sin\theta \;+\; f_y r\cos\theta$$

Use your computations above to show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2$$

Ans: From the first part above we get

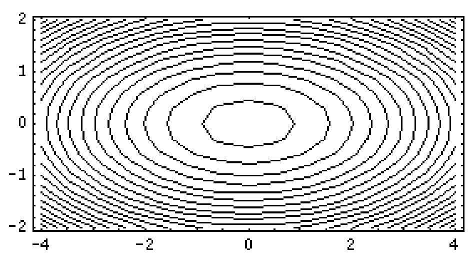
$$\begin{aligned}
\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 &= (f_x \cos\theta \,+\, f_y \sin\theta)^2 \,+\, \frac{1}{r^2}(-f_x r \sin\theta \,+\, f_y r\cos\theta)^2 \\
&= (f_x \cos\theta \,+\, f_y \sin\theta)^2 \,+\, (-f_x \sin\theta \,+\, f_y \cos\theta)^2 \\
&= f_x^2(\cos^2\theta + \sin^2\theta) \,+\, f_y^2(\cos^2\theta + \sin^2\theta) \,+\, 2 f_x f_y(\cos\theta\sin\theta - \cos\theta\sin\theta) \\
&= f_x^2 \,+\, f_y^2
\end{aligned}$$

Q3]... Suppose that the temperature, $T(x, y)$ in degrees Celsius, at the point (x, y) on a metal plate is given by

$$T(x, y) \;=\; 30e^{-(x^2+4y^2)}\,.$$

1. *Draw the level curves (isothermal lines) for T.*

   Ans: Note that $30e^{-u}$ is constant precisely when u is constant. Thus the level curves are a concentric family of ellipses of the form $x^2+4y^2 = c$. Their x-diameters will be twice as long as their y-diameters.

2. *Compute the gradient vector $\nabla T(x, y)$.*

   Ans: $\nabla T(x, y) = \langle T_x, T_y\rangle = \langle -60xe^{-(x^2+4y^2)}, -240ye^{-(x^2+4y^2)}\rangle$.

3. *Compute the directional derivative $D_{\hat{\mathbf{u}}}T(1, 1)$ where $\hat{\mathbf{u}}$ is the unit vector in the direction of $\langle 1, 2\rangle$.*

   Ans:

$$\begin{aligned}
D_{\hat{\mathbf{u}}}T(1,1) &= \nabla T(1,1)\cdot \hat{\mathbf{u}} \\
&= \langle -60e^{-5}, -240e^{-5}\rangle \cdot \langle 1/\sqrt{5}, 2/\sqrt{5}\rangle \\
&= \frac{-60(1) - 240(2)}{e^5\sqrt{5}} = -\frac{540}{e^5\sqrt{5}}
\end{aligned}$$

4. *Suppose an ant is about to walk away from the point $(1, 1)$ at unit speed. In which direction (on the plane) should the ant walk in order to experience the maximum rate of increase of temperature? [Your answer will be a vector].*

   Ans: The ant will experience the maximum rate of increase of temperature if it walks in the $\nabla T(1, 1) = \langle -60e^{-5}, -240e^{-5}\rangle$ direction. This is just the $\langle -1, -4\rangle$ direction.
