PRINT NAME: <u>SOLUTIONS</u>

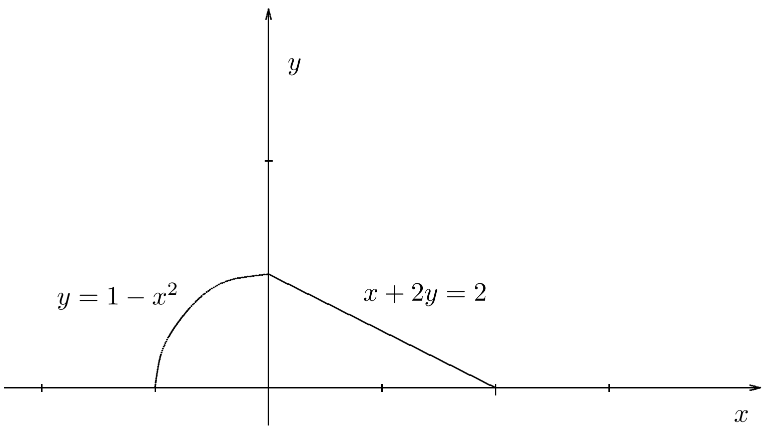
Calculus IV [2443–002] Midterm II

Q1]...[10 points] Consider the double integral

$$\int_0^1 \int_{-\sqrt{1-y}}^{2-2y} f(x,y)\,dx\,dy$$

Sketch the region of integration.

Soln. The limits $x = 2-2y$ and $x = -\sqrt{1-y}$ tell us that the region is bounded on the right by the line $x+2y=2$ and on the left by the parabola (left half) $y = 1-x^2$. The limits $y=0$ and $y=1$ tell us the upper and lower bounds for this region. We see that the parabola and line already intersect at $y=1$, so the region is drawn as shown.



Reverse the order of integration.

Soln. Note that reversing the order of integration means building up the region using vertical strips. There are two different tops on this region; the parabola top on the left side of the y-axis, and the straight line top on the right side. Thus, we have to divide the region into two pieces along the y-axis. So our answer will be a sum of two iterated integrals as shown.

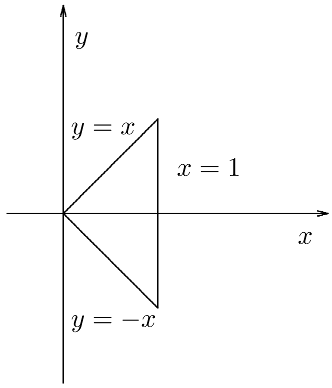
$$\int_{-1}^{0}\int_{0}^{1-x^2} f(x,y)\,dy\,dx \;+\; \int_0^2\int_0^{(2-x)/2} f(x,y)\,dy\,dx$$

Q2]...[10 points] Consider the following polar coordinates double integral

$$\int_{-\pi/4}^{\pi/4}\int_0^{\sec\theta} r^3 dr\,d\theta$$

Sketch the region of integration.

Soln. Note that the lines $\theta = \pi/4$ and $\theta = -\pi/4$ correspond to the cartesian lines $y = x$ and $y = -x$ respectively. Also the curve $r = \sec\theta$ is just $r = \frac{1}{\cos\theta}$ which rewrites as $r\cos\theta = 1$ or $x = 1$. [Gotta hate those trig functions!] Thus we get the following region.



Rewrite the integral as an iterated Cartesian coordinates integral (with appropriate limits). You do **NOT** have to compute this integral.

Soln. Remember that $dA = rdrd\theta$ and that the remaining r^2 can be written as $r^2 = x^2 + y^2$. Thus we get the following iterated integral.

$$\int_0^1 \int_{-x}^{x} x^2 + y^2 \, dy \, dx$$

Q3]...[20 points] Use double integrals in polar coordinates to compute the surface area of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ which is above the xy-plane and which lies inside the cone $z^2 = x^2 + y^2$. Your answer will involve a.

Soln. We saw in class that surface area of a portion (over the region R) of the graph of $z = f(x, y)$ is given by

$$\iint_R \sqrt{f_x^2 + f_y^2 + 1} \, dA$$

In this case we have (by implicit differentiation of the sphere equation) $f_x = -x/z$ and (likewise) $f_y = -y/z$. We did a sphere example in class! Check out the implicit differentiation details there! Thus,

$$f_x^2 + f_y^2 + 1 \; = \; \frac{x^2 + y^2 + z^2}{z^2} \; = \; \frac{a^2}{z^2}$$

which becomes

$$\frac{a^2}{a^2 - r^2}$$

in polar coordinates.

Now the cone equation is $z^2 = r^2$ and this intersects the sphere when $r^2 + r^2 = a^2$ or $r = a/\sqrt{2}$. Thus, the region that we are integrating over in the plane is given by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq a/\sqrt{2}$.

Filling all this into the surface area integral (and remembering that $dA = rdr\, d\theta$) gives

$$\int_0^{2\pi} \int_0^{a/\sqrt{2}} \frac{ar}{\sqrt{a^2 - r^2}} \, dr \, d\theta$$

We use the substitution $u = a^2 - r^2$ (so that $rdr = -du/2$) to evaluate the r integral. Here's what we end up with.

$$[\theta]_0^{2\pi} [-a\sqrt{a^2 - r^2}]_0^{a/\sqrt{2}} \; = \; (2\pi)(-a^2/\sqrt{2} + a^2) \; = \; 2\pi a^2(1 - 1/\sqrt{2})$$

Q4]...[20 points] Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = xy + z^2$ on the sphere $x^2 + y^2 + z^2 = 4$.

Soln. $\nabla f = \langle y, x, 2z\rangle$ and $\nabla g = \langle 2x, 2y, 2z\rangle$ so the Lagrange multiplier equations are

$$\begin{aligned} y &= 2\lambda x \\ x &= 2\lambda y \\ 2z &= 2\lambda z \\ 4 &= x^2 + y^2 + z^2 \end{aligned}$$

So that's it for the calculus. Now we just have to keep our head with all this algebra. First of all, note that the first two equations tell us that $x = 0$ precisely when $y = 0$ (since x and y are multiples of each other). So let's break this analysis into two cases.

Case I: [$x = 0$ **and** $y = 0$] In this case the fourth equation becomes $z^2 = 4$ and so we get $z = \pm 2$. Thus, we get two points: $(0, 0, 2)$ and $(0, 0, -2)$.

Case II: [$x \neq 0$ **and** $y \neq 0$] In this case the first two equations give $x/y = 2\lambda = y/x$. But this means that 2λ must be equal to its own reciprocal (since x/y and y/x are reciprocals) and so must be ± 1. Thus the third equation becomes $2z = \pm z$ which implies that $z = 0$. Now equation 4 becomes $2x^2 = 4$ or $x = \pm\sqrt{2}$. We get four points: $(-\sqrt{2}, -\sqrt{2}, 0)$, $(-\sqrt{2}, \sqrt{2}, 0)$, $(\sqrt{2}, -\sqrt{2}, 0)$, and $(\sqrt{2}, \sqrt{2}, 0)$.

Finally, we evaluate f on these 6 points and see that the maximum f-value is 4 (occurs at $(0, 0, 2)$ and $(0, 0, -2)$), and that the minimum f-value is -2 (occurs at $(-\sqrt{2}, \sqrt{2}, 0)$ and $(\sqrt{2}, -\sqrt{2}, 0)$).

Bonus Question. Let u and v be differentiable functions of one variable with derivatives denoted by u' and v' respectively. Let R be the triangular region with vertices at the points (a, a), (b, b) and (b, a). By evaluating the double integral

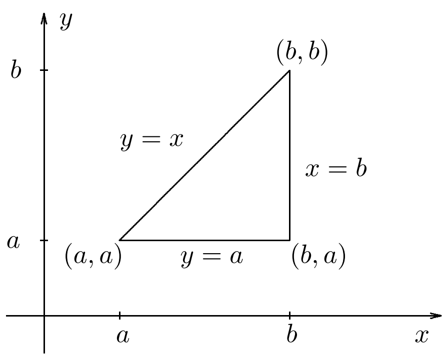
$$\iint_R u'(x)\, v'(y)\, dA$$

in two different ways (as iterated integrals), give a new derivation of the integration by parts formula

$$\int_a^b u\, dv = uv\,\Big|_a^b - \int_a^b v\, du\,.$$

Soln.

Here is a diagram of the triangular region with its sides labelled.



On the one hand, we can integrate with respect to y first to get

$$\begin{aligned}
\iint_R u'(x)\,v'(y)\,dA &= \int_a^b \left(\int_a^x u'(x)v'(y)\,dy\right)\,dx \\
&= \int_a^b [u'(x)v(y)]_a^x\,dx \\
&= \int_a^b u'(x)v(x) - u'(x)v(a)\,dx \\
&= \int_a^b v(x)u'(x)\,dx \;-\; [u(x)v(a)]_a^b \\
&= \int_a^b v(x)u'(x)\,dx \;-\; u(b)v(a) \;+\; u(a)v(a)
\end{aligned}$$

On the other hand, we can integrate with respect to x first to get

$$\begin{aligned}
\iint_R u'(x)\,v'(y)\,dA &= \int_a^b \left(\int_y^b u'(x)v'(y)\,dx\right)\,dy \\
&= \int_a^b [u(x)v'(y)]_y^b\,dy \\
&= \int_a^b u(b)v'(y) - u(y)v'(y)\,dy \\
&= [u(b)v(y)]_a^b \;-\; \int_a^b u(y)v'(y)\,dy \\
&= u(b)v(b) \;-\; u(b)v(a) \;-\; \int_a^b u(y)v'(y)\,dy
\end{aligned}$$

Finally, setting these two expressions equal to each other gives

$$\int_a^b v(x)u'(x)\,dx \;-\; u(b)v(a) \;+\; u(a)v(a) \;=\; u(b)v(b) \;-\; u(b)v(a) \;-\; \int_a^b u(y)v'(y)\,dy$$

which simplifies down to (removing explicit reference to the dummy variables of integration x and y)

$$\int_a^b v\,du \;=\; uv|_a^b \;-\; \int_a^b u\,dv$$

and we're finished.