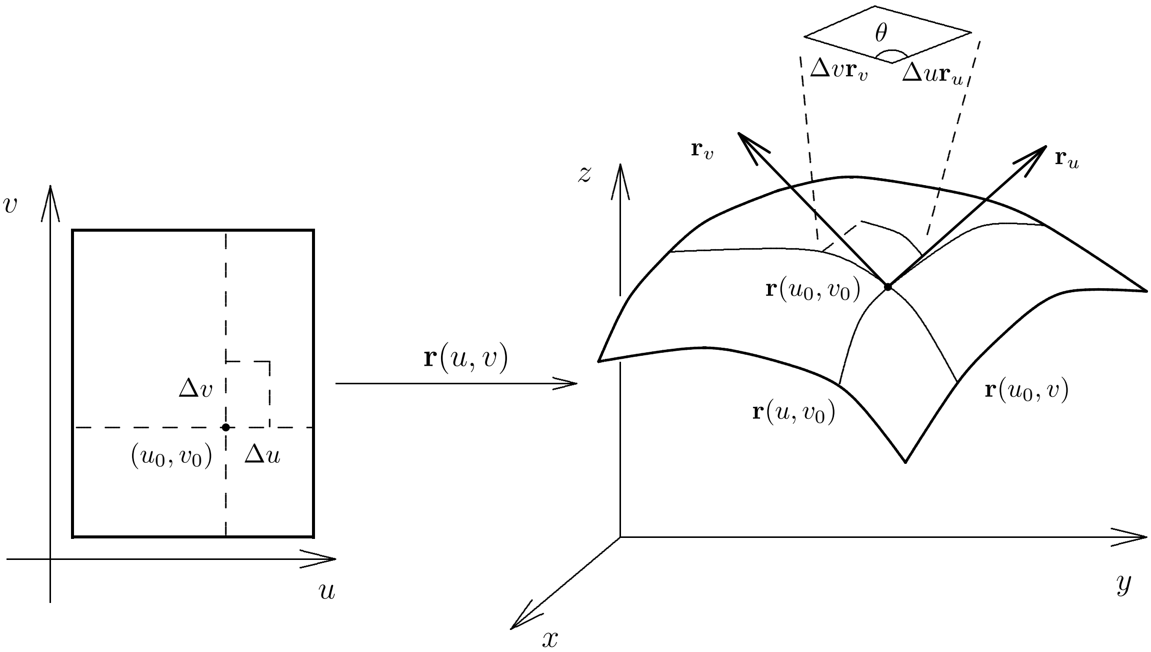
Math 2443-002 Calculus IV Spring 2000

Surface Integrals

We begin with a discussion of surface area for parametric surfaces.



The parametric map $\mathbf{r}(u, v)$ takes a rectangle of side lengths Δu and Δv to a curvilinear *patch* on the parametric surface in 3-d as shown in the diagram. The area of this patch is approximated (up to first order) by the area of a parallelogram in 3-d with sides given by the tangent vectors $\Delta u\mathbf{r}_u$ and $\Delta v\mathbf{r}_v$. This area is just

$$|\mathbf{r}_u||\mathbf{r}_v| \sin\theta \Delta u \Delta v$$

which is equal to (by Calculus III – cross products)

$$|\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v \,.$$

Taking Riemann sums gives us a surface area integral

$$\iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$$

where D is a region in the uv-plane. We show how to compute this as a double integral on the next page, and we define surface integrals (over a parametric surface) for functions or vector fields.

Surface Area: Let S be a parametric surface given by the vector equation

$$\mathbf{r}(u,v) \;=\; \langle\, x(u,v),\; y(u,v),\; z(u,v)\,\rangle$$

where (u,v) belong to a region D in the uv-plane. Then the surface area of S is computed as

$$\begin{aligned} Area(S) &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dudv \\ &= \iint_D |\langle \tfrac{\partial(y,z)}{\partial(u,v)}, \tfrac{\partial(z,x)}{\partial(u,v)}, \tfrac{\partial(x,y)}{\partial(u,v)}\rangle|\, dudv \\ &= \iint_D \sqrt{\left(\tfrac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\tfrac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\tfrac{\partial(x,y)}{\partial(u,v)}\right)^2}\, dudv \end{aligned}$$

Note that when the surface is just the graph of $z = f(x,y)$ the expression above reduces to

$$\iint_D \sqrt{1 + (f_x)^2 + (f_y)^2}\, dxdy$$

which is a nice generalization of the formula $\int \sqrt{1 + (f'(x))^2} dx$ for arclength along the graph of $y = f(x)$.

Surface Integrals: One can define surface integrals of a function or a vector field as follows.

If $f(x,y,z)$ is a continuous function defined on a region of $\mathbf{R}^3$ which contains the surface S, then we define

$$\iint_S f\, ds \;=\; \iint_D f(\mathbf{r}(u,v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dudv\,.$$

If $\mathbf{F} = \langle\, P, Q, R\,\rangle$ is a continuous vector field defined on a region of $\mathbf{R}^3$ which contains the surface S, then we define

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{s} &= \iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})\, ds \\ &= \iint_D \mathbf{F}(\mathbf{r}(u,v))\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dudv\,. \end{aligned}$$

Note that the last term above is just an iterated integral. There's quite a bit of computation here, and it takes a bit of practice to get proficient at converting a surface integral into a standard iterated (double) integral in u and v.

Remark: When the surface is just the graph of $z = f(x,y)$, $x, y \in D$, then we get

$$\iint_S \mathbf{F}\cdot ds \;=\; \iint_D \mathbf{F}\cdot\langle\, -f_x, -f_y, 1\,\rangle\, dxdy \;=\; \iint_D (R - Qf_y - Pf_x)\, dxdy\,.$$

Flux: When $\mathbf{F}$ is the velocity vector field of a fluid flow, then $\iint_S \mathbf{F}\cdot d\mathbf{s}$ represents the net fluid flowing through the surface S per unit time. It is called the *flux of* $\mathbf{F}$ *across* S.