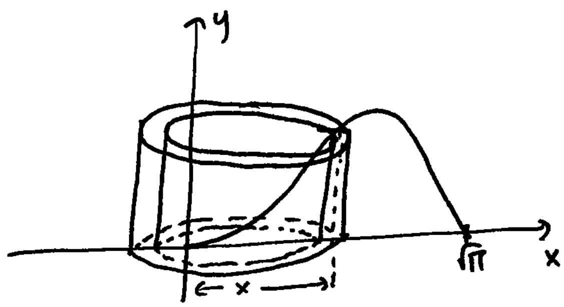
Homework #5

Section 6.3

2)



Cylindrical shell circumference $= 2\pi x$

Height $= \sin(x^2)$

$$V = \int_0^{\sqrt{\pi}} 2\pi x \cdot \sin(x^2) \cdot dx$$

Let $u = x^2$ then $du = 2x\,dx$

For $x = \sqrt{\pi}$, $u = \pi$

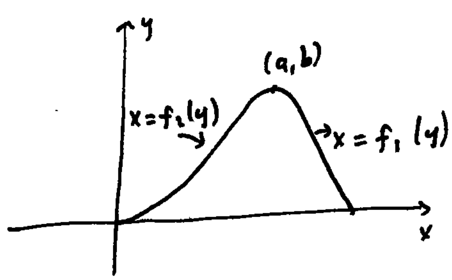
For $x = 0$, $u = 0$

hence $$V = \int_{x=0}^{x=\sqrt{\pi}} 2\pi x \cdot \sin(x^2) \cdot dx = \int_{u=0}^{u=\pi} \pi \cdot \sin u\, du = \pi \int_0^{\pi} \sin u\, du = \pi \cdot [-\cos u]_0^{\pi}$$

$$= \pi[-\cos\pi - (-\cos 0)]$$

$$= \pi[-(-1) - (-1)]$$

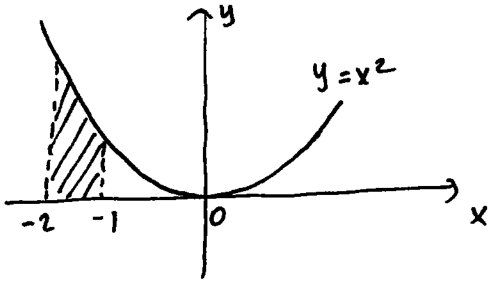
$$= 2\pi$$

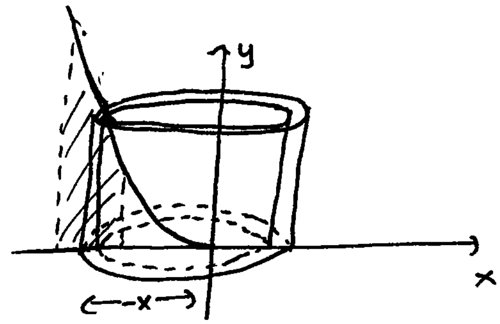


For slicing, we need to find the local maximum point of the given function. when we find the local maximum we seperate the curve $\sin(x^2) = y$ into two pieces as $x = f_1(y)$ and $x = f_2(y)$ which are on the right handside and lefthandside on the graph of (a,b) respectively

Hence $$V = \pi \int_0^b [f_1(y)]^2 - [f_2(y)]^2]\,dy$$. Using shells is definitely preferable to slicing.

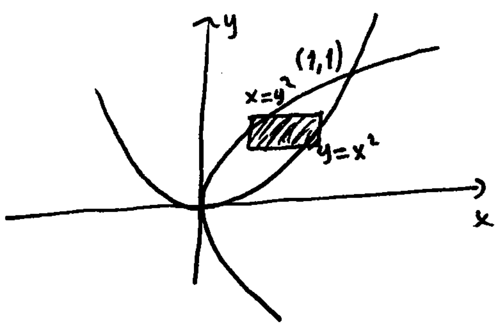
16)

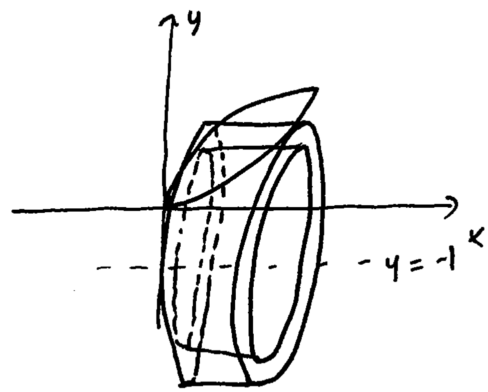




$$V = \int_{-2}^{-1} 2\pi(-x) . x^2 \, dx = 2\pi \left[-\frac{1}{4} x^4 \right]_{-2}^{-1} = \frac{15}{2}\pi .$$

20)





$$V = \int_0^1 2\pi \, (y+1) \, (\sqrt{y} - y^2) \, dy = 2\pi \int_0^1 (y^{3/2} + y^{1/2} - y^3 - y^2) dy = \frac{29\pi}{30}$$

38) $$V = \int_1^2 2\pi x \, (-x^2 + 3x - 2) \, dx = 2\pi \int_1^2 (-x^3 + 3x^2 - 2x) \, dx = \frac{\pi}{2}$$

by using shells.

40) By washers method

$$V = \int_{-1}^{1} \pi \left[(2-0)^2 - [2-(1-y^4)]^2 \right] dy = 2 \int_0^1 \pi \left[4 - (1+y^4)^2 \right] dy \text{ (by symmetry)}$$

$$= \frac{224\pi}{45}$$

42) By shells method

$$V = \int_0^2 2\pi y \left[\sqrt{1-(y-1)^2} - \left(-\sqrt{1-(y-1)^2}\right) \right] dy$$

$$= 2\pi \int_0^2 y \cdot 2 \sqrt{1-(y-1)^2}\, dy$$

Let $u = y-1$ then $du = dy$ hence

$$= 4\pi \int_{u=-1}^{u=1} u\sqrt{1-u^2}\, du + 4\pi \int_{-1}^{1} \sqrt{1-u^2}\, du$$

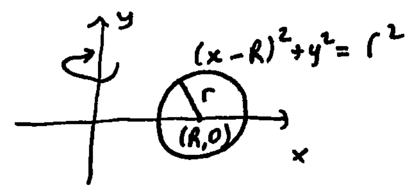
$\int_{-1}^{1} u\sqrt{1-u^2}\, du = 0$ since its integrand is an odd function

$\int_{-1}^{1} \sqrt{1-u^2}\, du = \frac{\pi}{2}$ since it is the area of a semicircle or radius 1.

hence

$$V = 4\pi \cdot 0 + 4\pi \cdot \frac{\pi}{2} = 2\pi^2$$

44) $$V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x-R)^2}\, dx$$



Let $u = x-R$ then $du = dx$ so

$$V = \int_{-r}^{r} 4\pi (u+R) \cdot \sqrt{r^2-u^2}\, du = 4\pi R \int_{-r}^{r} \sqrt{r^2-u^2}\, du + 4\pi \int_{-r}^{r} u \cdot \sqrt{r^2-u^2}\, du.$$

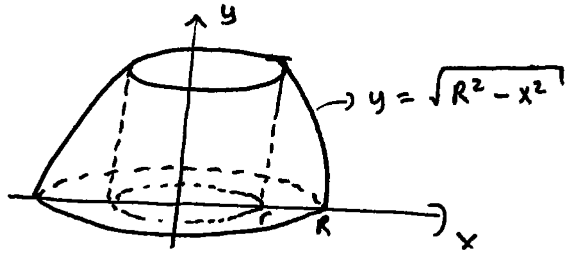
$\int_{-r}^{r} \sqrt{r^2-u^2}\, du = \frac{1}{2}\pi r^2$ since it is the area of a semicircle with radius r.

$\int_{-r}^{r} u \cdot \sqrt{r^2-u^2}\, du = 0$ since the integrand is an odd function

So

$$V = 4\pi R\left(\tfrac{1}{2}\pi r^2\right) + 4\pi \cdot 0 = 2\pi R . r^2$$

46)



By symmetry

$$V = 2\int_{\text{inner radius}}^{\text{outer radius}} 2\pi r h \, dx = 2\int_r^R 2\pi . x\sqrt{R^2 - x^2}\, dx$$

$$= 4\pi \int_r^R x . \sqrt{R^2 - x^2}\, dx$$

Let $u = R^2 - x^2$ then $du = -2x\,dx \Rightarrow x\,dx = \frac{du}{-2}$

$$= 4\pi \int_{x=r}^{x=R} u^{1/2} \cdot \frac{du}{-2} = -2\pi\left[\tfrac{2}{3} u^{3/2}\right]_{x=r}^{x=R}$$

$$= -2\pi\left[\tfrac{2}{3}(R^2 - x^2)^{3/2}\right]_{x=r}^{x=R}$$

$$= \frac{4}{3}\pi (R^2 - r^2)^{3/2}$$

By Pythagorean theorem,

$$R^2 = \left(\tfrac{1}{2}h\right)^2 + r^2$$

so the volume of the napkin ring is $\frac{4}{3}\pi\left(\frac{1}{2}h\right)^3 = \frac{1}{6}\pi h^3$ which is independent of both R and r, so the amount of wood in a ring of height h is the same regardless of the site of the sphere used

$$V_{\text{napkin ring}} = V_{\text{sphere}} - V_{\text{cylinder}} - 2V_{\text{cap}}$$

$$= \frac{4}{3}\pi R^3 - \pi r^2 h - 2\cdot\frac{\pi}{3}\left(R - \frac{1}{2}h\right)^2\left[3R - \left(R - \frac{1}{2}h\right)\right] = \frac{1}{6}\pi h^3$$

where height of cap is $R - \frac{1}{2}h$.