SPRING '07 — CALC II — MIDTERM I (SOLUTIONS)

Q1]...[13 points] Write down the Riemann sum for the function x^2 over the interval $[1,3]$ using 4 equal width subintervals, and using right hand endpoints as evaluation points. You can leave your answer as a string of numbers with plus signs between them.

$\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$ so end points are: $1, \; 3/2, \; 2, \; 5/2, \; 3$

$$R_4 = \left(\tfrac{3}{2}\right)^2\left(\tfrac{1}{2}\right) + (2)^2\left(\tfrac{1}{2}\right) + \left(\tfrac{5}{2}\right)^2\left(\tfrac{1}{2}\right) + (3)^2\left(\tfrac{1}{2}\right)$$

Write the following limit as a definite integral

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \sin\left(1+\frac{3i}{n}\right)$$

$$\int_1^4 \frac{2}{3}\sin(x)\,dx$$

Interval starts at 1

& ends at $1 + \frac{3\cancel{n}}{\cancel{n}} = 1+3 = 4$

$\Rightarrow [1,4]$.

$\Delta x = \frac{4-1}{n} = \frac{3}{n}$

$\Delta x \cdot f(x) \Rightarrow f(x) = \frac{2}{3}\sin(x)$

$\frac{2}{n}\sin(\ldots) = \left(\frac{3}{n}\right)\frac{2}{3}\sin(\ldots)$

Compute the average value of the function $f(x) = x^2$ on the interval $[0,4]$.

$$\hat{f} = \frac{1}{4-0}\int_0^4 f(x)\,dx$$

$$= \frac{1}{4}\int_0^4 x^2\,dx$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \boxed{\frac{16}{3}}$$

Q2]... **[12 points]** State the fundamental theorem of calculus (both parts).

$f(x)$ continuous on $[a,b]$, then

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x).$$

$F'(x) = f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Evaluate the following definite integral

$$\int_0^{\pi} x + 2\sin(x)\,dx$$

$$= \left.\frac{x^2}{2} - 2\cos(x)\right]_0^{\pi} = \left(\frac{\pi^2}{2} - 2\cos(\pi)\right) - \left(\frac{0^2}{2} - 2\cos(0)\right)$$

by F.T.

$$= \pi^2/2 + 2 + 2 = \boxed{\pi^2/2 + 4}$$

Evaluate the following derivative

$$\frac{d}{dx}\left(\int_{x^2}^{7} \frac{\sin(t)}{t}\,dt\right)$$

$$= -\frac{d}{dx}\left(\int_7^{x^2} \frac{\sin(t)}{t}\,dt\right)$$

$$= \boxed{-2x\,\frac{\sin(x^2)}{x^2}}$$ by F.T. & Ch. Rule.

Q3]... **[12 points]** Determine the following indefinite integrals.

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x} = x^{\frac{1}{2}}$

$\Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\Rightarrow 2du = \frac{dx}{\sqrt{x}}$

$\Rightarrow \int = \int \sin(u) \cdot 2\,du$

$= -2\cos(u) + C$

$= \underline{-2\cos(\sqrt{x}) + C}$

$$\int \sin^3(2x)\cos(2x)\,dx$$

Let $u = \sin(2x)$

$\Rightarrow \frac{du}{dx} = \cos(2x) \cdot 2$

$\Rightarrow \frac{du}{2} = \cos(2x)dx$

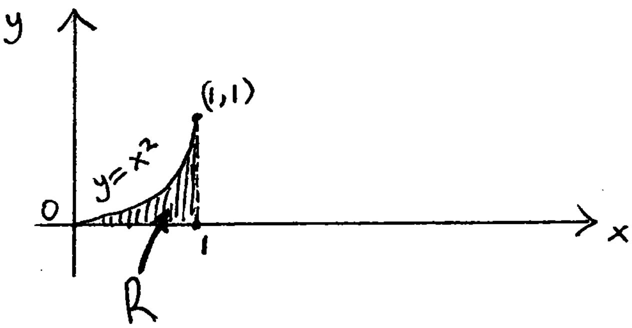
$\Rightarrow \int = \int u^3 \frac{du}{2}$

$= \frac{1}{2}\frac{u^4}{4} + C$

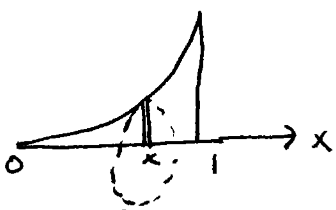
$= \underline{\frac{1}{8}\sin^4(2x) + C}$

Q4]...[13 points] In each case below, use the disk or washer method to write down an integral for the volume of revolution of the given region about the given line. You do **not** have to evaluate the resulting integrals.

$R =$ Region below $y = x^2$, between 0 & 1.



- The region R about the x-axis.

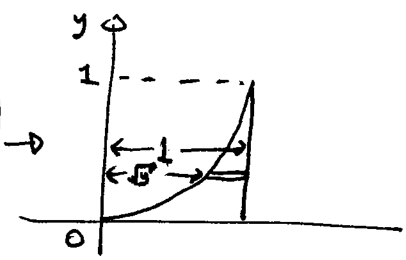


Area $= \pi(\text{rad})^2 = \pi (x^2)^2$

$$\Rightarrow \text{Vol} = \int_0^1 \pi (x^2)^2 \, dx$$

- The region R about the y-axis.

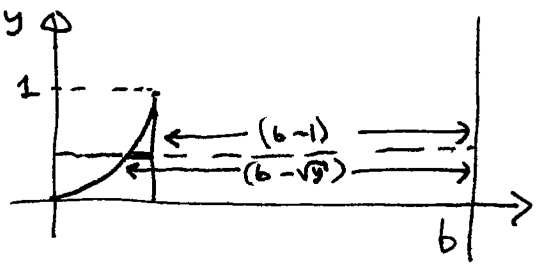
$y = x^2$
$x = \sqrt{y}$



Area $= \pi (1)^2 - \pi (\sqrt{y})^2$

$$\text{Vol} = \int_0^1 \pi \left(1^2 - (\sqrt{y})^2\right) dy$$

- The region R about the line $x = 6$.



Note $\boxed{x=6}$ is a vertical line!

Area $= \pi (6 - \sqrt{y})^2 - \pi (6-1)^2$

$$\text{Vol} = \pi \int_0^1 \left[(6 - \sqrt{y})^2 - 5^2\right] dy$$

Comments: → You should remember "+C" for indefinite integrals

→ Present your work clearly + cleanly, showing the "steps" (= thought processes) you have taken. Simply writing down answers (even if correct) will only earn you a small % of total points.

→ Algebra errors: $\pi(a-b)^2 \neq \pi(a^2-b^2)$

which are you likely to see in a "washer" method computation?

→ Watch for calc I errors: (Particularly in "substitution" problems)

eg: $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{\frac{1}{2}-1}$

$= \frac{1}{2}x^{-\frac{1}{2}}$

(Many people got $x^{-3/2}$)

$= \frac{1}{2\sqrt{x}}$

is correct, but ...

→ Washer/Disk method ⇒ you slice perpendicular to axis of revolution: vertical axis ⟷ dy integral

horizontal axis ⟷ dx integral.