

SOLUTIONS - CALC II - SP'08

Q1]... [10 points] Evaluate the following limit by first converting it into a suitable form for l'Hospital's rule.

$$\lim_{x \rightarrow 1^+} \left( \frac{x}{x-1} - \frac{1}{\ln x} \right)$$

$$= \lim_{x \rightarrow 1^+} \left( \frac{x \ln(x) - 1(x-1)}{(x-1) \ln(x)} \right) \quad \leftarrow \text{this is in } \frac{0}{0} \text{ form!}$$

$$= \lim_{x \rightarrow 1^+} \left( \frac{1 \ln(x) + \frac{x}{x} - 1}{1 \ln(x) + \left( \frac{x-1}{x} \right)} \right)$$

L'Hôpital's rule (% form)

$$= \lim_{x \rightarrow 1^+} \left( \frac{\ln(x)}{\ln(x) + \frac{x-1}{x}} \right)$$

$$= \lim_{x \rightarrow 1^+} \left( \frac{x \ln(x)}{x \ln(x) + (x-1)} \right)$$

$$= \lim_{x \rightarrow 1^+} \left( \frac{1 \ln(x) + \frac{x}{x}}{1 \ln(x) + \frac{x}{x} + 1} \right) = \frac{0+1}{0+1+1}$$

L'Hôpital's rule (% form)

$$= \boxed{\frac{1}{2}}$$

Q2]... [10 points] Evaluate the indefinite integral

$$\int \sin^4(x) dx$$

$$= \int (\sin^2(x))^2 dx$$

$$= \int \left( \frac{1 - \cos(2x)}{2} \right)^2 dx$$

$$= \frac{1}{4} \int 1 + \cos^2(2x) - 2 \cos(2x) dx$$

$$= \frac{1}{4} \int 1 + \frac{1 + \cos(4x)}{2} - 2 \cos(2x) dx$$

$$= \frac{1}{4} \left[ \frac{3x}{2} + \frac{\sin(4x)}{8} - \sin(2x) \right] + C$$

Q3]... [10 points] Evaluate the following improper integral

$$\int_e^{\infty} \frac{\ln(x) dx}{x^{10}}$$

$$= \lim_{T \rightarrow \infty} \left( \int_e^T \frac{\ln(x)}{x^{10}} dx \right)$$

$$= \lim_{T \rightarrow \infty} \left[ \frac{-\ln(x)}{9x^9} \Big|_e^T + \frac{1}{9} \int_e^T \frac{dx}{x^{10}} \right]$$

Int by parts

$$u = \ln(x) \rightarrow du = \frac{dx}{x}$$

$$dv = x^{-10} dx \rightarrow v = -\frac{1}{9} x^{-9}$$

$$= \lim_{T \rightarrow \infty} \left( \frac{1}{9e^9} - \frac{\ln(T)}{9T^9} - \frac{1}{81} \frac{1}{x^9} \Big|_e^T \right)$$

$$= \lim_{T \rightarrow \infty} \left( \frac{1}{9e^9} - \frac{\ln(T)}{9T^9} + \frac{1}{81e^9} - \frac{1}{81T^9} \right)$$

$$= \frac{1}{9e^9} + \frac{1}{81e^9}$$

Q4]... [10 points] Find an expression for the inverse of the following function

$$y = \frac{10^x - 10^{-x}}{2}$$

①  $x \leftrightarrow y$ :

$$x = \frac{10^y - 10^{-y}}{2}$$

② solve for "new y":

$$2x = 10^y - \frac{1}{10^y}$$

$$(10^y)^2 - 2x(10^y) - 1 = 0$$

Quadratic

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow 10^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$10^y = \frac{2x + \sqrt{4x^2 + 4}}{2}$$

$$10^y = x + \sqrt{x^2 + 1}$$

Take  $+\sqrt{\quad}$   
Since  $10^y > 0$

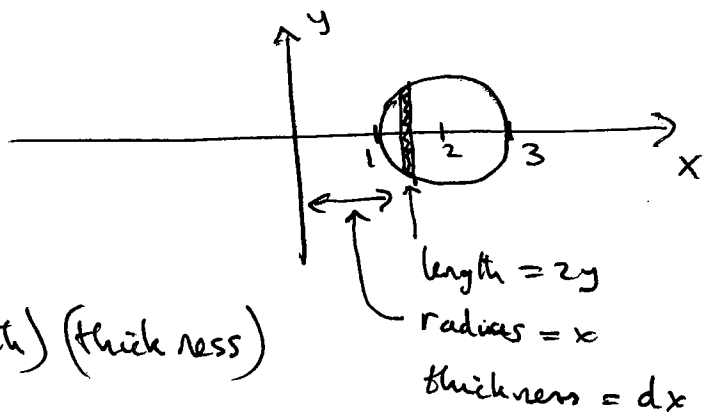
$$y \ln(10) = \ln(10^y) = \ln(\sqrt{x^2 + 1} + x)$$

$$y = \frac{\ln(\sqrt{x^2 + 1} + x)}{\ln(10)}$$

Q5]. . . [10 points] Using the SHELL method, determine the volume (of the "bagel") obtained by rotating the region inside the circle

$$(x-2)^2 + y^2 = 1$$

about the y-axis.



$$V \Delta x = \int 2\pi (\text{radius}) (\text{length}) (\text{thickness})$$

$$= \int_1^3 2\pi x (2y) dx$$

$$= \int_1^3 2\pi x 2 \sqrt{1-(x-2)^2} dx$$

$$= 4\pi \int_1^3 x \sqrt{1-(x-2)^2} dx$$

Trig Subst  $\left\{ \begin{array}{l} \text{Let } (x-2) = \sin \theta \\ dx = \cos \theta d\theta \\ \sqrt{\quad} = \cos \theta \\ \text{limits are } \pm \pi/2 \end{array} \right.$

$$= 4\pi \int_{-\pi/2}^{\pi/2} (2 + \sin \theta) \cos \theta \cos \theta d\theta$$

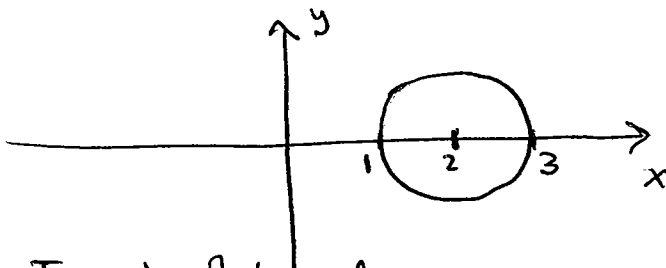
$$= 8\pi \int_{-\pi/2}^{\pi/2} \cos^2 \theta d\theta + 4\pi \int_{-\pi/2}^{\pi/2} \sin \theta \cos^2 \theta d\theta = 0 \quad (\text{odd integrand!})$$

$$= 8\pi \int_{-\pi/2}^{\pi/2} \left( \frac{1 + \cos(2\theta)}{2} \right) d\theta = 4\pi \left[ \theta + \frac{\sin(2\theta)}{2} \right]_{-\pi/2}^{\pi/2} = \boxed{4\pi^2}$$

Q6]. . . [15 points] Determine the surface area (of the "bagel") obtained by rotating the circle

$$(x-2)^2 + y^2 = 1$$

about the  $y$ -axis.



Surface Area = 2 Surface Area of Top  $\frac{1}{2}$  of bagel.

$$= 2 \int 2\pi (\text{radius}) ds$$

$$= 2 \int 2\pi x ds$$

$$= 4\pi \int_1^3 x \sqrt{1+(y')^2} dx$$

$$= 4\pi \int_1^3 \frac{x}{y} dx$$

$$= 4\pi \int_1^3 \frac{x dx}{\sqrt{1-(x-2)^2}}$$

$$= 4\pi \int_{-\pi/2}^{\pi/2} \frac{(2+\sin\theta) \cancel{\cos\theta} d\theta}{\cancel{\cos\theta}}$$

$$= 4\pi \int_{-\pi/2}^{\pi/2} 2 d\theta + 4\pi \int_{-\pi/2}^{\pi/2} \sin\theta d\theta \rightarrow 0 \text{ add integrand}$$

$$= \boxed{8\pi^2}$$

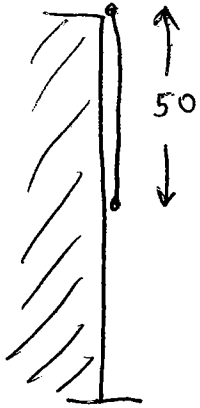
$$\begin{aligned} (x-2)^2 + y^2 &= 1 \\ \downarrow \\ 2(x-2) + 2yy' &= 0 \\ \downarrow \\ y' &= -\left(\frac{x-2}{y}\right) \\ \sqrt{1+(y')^2} &= \sqrt{\frac{y^2 + (x-2)^2}{y^2}} \\ &= \frac{1}{y} \end{aligned}$$

← Trig Subst  $\left\{ \begin{array}{l} \text{As before} \\ (x-2) = \sin\theta \\ dx = \cos\theta d\theta \\ \sqrt{\phantom{x}} = \cos\theta \\ \text{Limits} = \pm \pi/2 \end{array} \right.$

Q7]... [10 points] Write down the work done when the constant force  $F$  lbs moves an object through a distance  $d$  ft.

$$\text{Work} = (\text{force})(\text{distance}) = F \cdot d \quad \text{ft-lbs}$$

A uniform 50 ft long rope weighing 2 lb/ft hangs over the edge of a tall building as shown. Using Riemann sums, show how to obtain an integral expression for the work done in lifting the rope to the top of the building. Give reasons for your steps.



Chop rope into  $n$  equal length pieces, each of length  $\Delta x = \frac{50}{n}$

$i$ th piece has weight  $= 2 \Delta x = 2 \Delta x_i$   
 $i$ th piece will travel height of  $x_i^*$  (between  $x_{i-1}$  &  $x_i$ )  
 to top of building  
 $\Rightarrow$  work done  $= 2 x_i^* \Delta x_i$   
 $\Rightarrow$  Total work done for all  $n$  pieces  $= \sum_{i=1}^n 2 x_i^* \Delta x_i$

Now take limit as  $n \rightarrow \infty$  to get continuous version ...

$$\text{Work} = \lim_{n \rightarrow \infty} \sum_{i=1}^n 2 x_i^* \Delta x_i = \int_0^{50} 2x \, dx$$

What is the work done in lifting the rope above to the top of the building? That is, evaluate the integral you found above.

$$\text{Work} = \int_0^{50} 2x \, dx = \left[ \frac{2x^2}{2} \right]_0^{50} = 2500 \text{ ft-lbs.}$$

