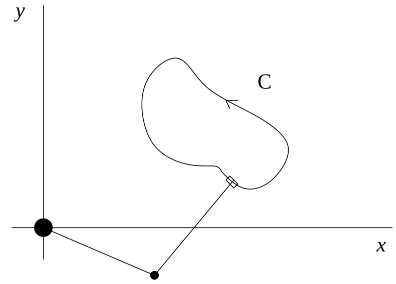
Geometric Applications of Green's and Divergence Theorems

Q1. Green's Theorem and Planimeters.



Here is a picture of a planimeter. It consists of two rigid arms (of equal length L say). The end of one arm is attached to a fixed hinge at the origin (it can swivel about the origin through 2π). The second arm is attached to the end of the first arm by a hinge which also allows for 2π rotations. See the figure. The end of the second arm has a wheel. This wheel will rotate if the second arm is moved in a perpendicular direction to itself, and the wheel will slide (and not rotate) if the arm is moved parallel to itself.

You have to show that the total amount of rotation of the wheel as we move the end of the second arm around a simple closed curve C is proportional to the area enclosed by C. Thus the planimeter can be used to compute the areas of planar regions with complicated boundaries. The only restriction is the size of the planimeter (the curve C must be contained in a disk of radius $2L$ about the origin in the plane).

(1) Define a unit vector field corresponding to the planimeter called the *rolling vector field* $\mathbf{R}$ as follows. The vector $\mathbf{R}$ is located at the end (wheel) of the planimeter. It has unit length and is orthogonal to the second arm vector of the planimeter. It points to the left of the second arm vector (the second arm vector points from the elbow to the wheel).

Find a formula for $\mathbf{R}(x, y)$. That is, suppose that the wheel is at point (x, y). Find the coordinates (a, b) of the elbow of the planimeter (you may assume planimeter is bent as shown throughout its use). Find the second arm vector, and finally find $\mathbf{R}$.

(2) Prove that as the wheel end of the planimeter is pushed along a segment Δs of arclength of the curve C, the amount of rolling of the wheel is given by

$$(\mathbf{T} \cdot \mathbf{R})(\Delta s)$$

where $\mathbf{T}$ is the unit tangent vector to the path C at some point along the Δs segment.

Conclude that the total amount of rolling of the planimeter wheel as we move around C is given by

$$\int_C \mathbf{R} \cdot d\mathbf{r}$$

(3) Use Green's theorem and the form of $\mathbf{R}$ in part 1 above, to argue that the total rotation is proportional to the area enclosed by C.

Q2. Divergence Theorem and Solid Angles.

Do Q1 on page 1145 of the textbook (this is the solid angle question in the Problems Plus section at the end of chapter 17).