Q1]. . . [10 points] Sketch the level curves $f = 1$, $f = 4$, $f = 0$, $f = -1$, and $f = -4$ of the function $f(x, y) = x^2 - 4y^2$.

Q2]. . . [10 points] Does the following limit exist? Give reasons for your answer.

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{x^2 + 2y^2}$$

Q3]. . . [12 points] The area of a triangular field of side lengths x, y and contained angle θ is given by $A = \frac{1}{2}xy\sin\theta$. Write down the differential dA, and use it to estimate the error in the area of a field with side measurements of 150m and 200m (each accurate to within ± 1m) and contained angle of 30° (accurate to within $\pm 2^\circ$).

Q4]. . . [12 points] Write down ∇f for the function $f(x, y, z) = x^2y^3z^4$.

In what direction is f increasing most rapidly at the point (2,1,3)?

What is the value of the maximum rate of change of f at the point $(2, 1, 3)$?

Q5]. . . [13 points] Prove that the two surfaces $x^2 + y^2 + z^2 = 25$ and $x^2 = 4y^2 + 4z^2$ are perpendicular (orthogonal) to each other at all points of intersection.

Q6]. . . [18 points] Suppose that $z = f(x, y)$ where $x = g(s, t)$ and $y = h(s, t)$. Verify that

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}$$

Show all the steps involved. Also, write down (no work needs to be shown) a similar expression for $\frac{\partial^2 z}{\partial s^2}$.

Bonus]. . . Suppose that $f(x, y)$ is differentiable at the point (a, b), and that you are told the values of the directional derivatives, $D_{\mathbf{u}}f(a, b)$ and $D_{\mathbf{v}}f(a, b)$, of f at the point (a, b) in the directions specified by the unit vectors $\mathbf{u}$ and $\mathbf{v}$. Suppose that $\mathbf{u}$ (resp. $\mathbf{v}$) makes an angle ϕ_u (resp. ϕ_v) with the positive x-axis. What (minimal) condition must ϕ_u and ϕ_v satisfy in order to reclaim the values of $f_x(a, b)$ and $f_y(a, b)$? Show how to compute the values of $f_x(a, b)$ and $f_y(a, b)$ from the two directional derivatives above.

Rough Work Page