MATH 4513-001 Senior Seminar Spring 2010

Homework Set 2

Please complete by class time on Thursday, Feb 11.

1. Consider the multiplicative group $G_1 = \{e^{n\pi i/4} \,|\, n \in \mathbb{Z}\}$.
   (a) Is it finite or infinite?
   (b) Draw it on the unit circle $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$.
   (c) Plot some of the elements of the multiplicative group $G_2 = \{(1+i)^n \,|\, n \in \mathbb{Z}\}$ in the complex plane (minus the origin) At least, plot the values of n between -7 and 9. Hint: it's is easier if you first write $(1+i)$ in polar form.
   (d) What is the image of G_2 under the homomorphism
   $$h : \mathbb{C} - \{0\} \to S^1 : z \mapsto \frac{z}{|z|}?$$

2. A group G is said to be *abelian* if the operation is *commutative.* That is $gh = hg$ for all $g, h \in G$. Show that the group $\mathrm{SL}(2, \mathbb{Z})$ is not abelian.

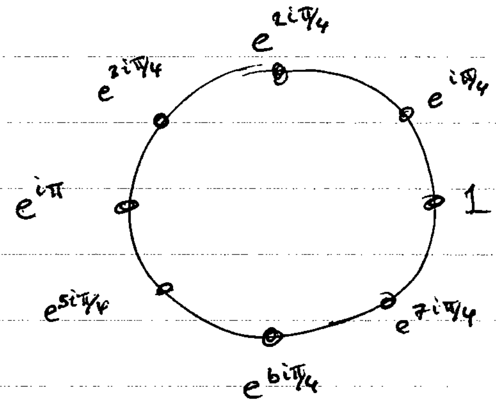
3. An $n \times n$ matrix A is said to be *orthogonal* if $A^T A = AA^T = I_n$, where I_n denotes the identity matrix (1's on the diagonal and 0's elsewhere) and A^T denotes the *transpose* of A (which is defined by requiring that its ij-entry is the ji-entry of A).
   (a) Check that the set $\mathrm{O}(n)$ of all $n \times n$ orthogonal matrices with real number entries is a group under matrix multiplication.
   (b) Prove that the elements of $\mathrm{O}(2)$ are either of the form
   $$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
   where $a^2 + b^2 = 1$.
   (c) Writing $a = \cos\theta$ and $b = \sin\theta$ above, give a geometric interpretation of the two types of elements of $\mathrm{O}(2)$.
   (d) Verify that $\mathrm{SO}(2) = \{A \in \mathrm{O}(2) \,|\, |A| = 1\}$ is a subgroup of $\mathrm{O}(2)$. It is called the *special orthogonal group.*

1(a). $G_1 = \{e^{n\pi i/4} \mid n \in \mathbb{Z}\}$ has 8 elements.

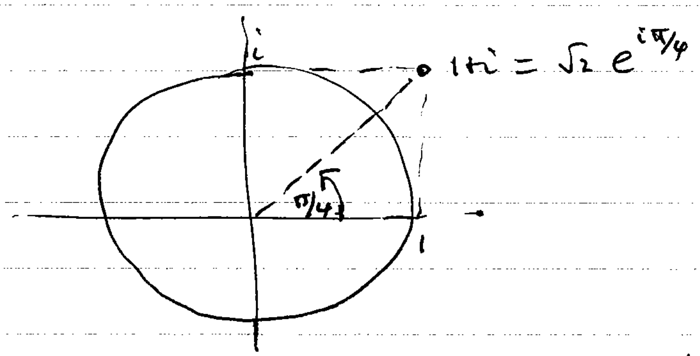
$$= \{e^{i\pi/4}, e^{2i\pi/4}, e^{3i\pi/4}, e^{i\pi}, e^{5i\pi/4}, e^{6i\pi/4}, e^{7i\pi/4}, 1\}$$

$= e^{8i\pi/4} = e^{2\pi i}$

1(b)

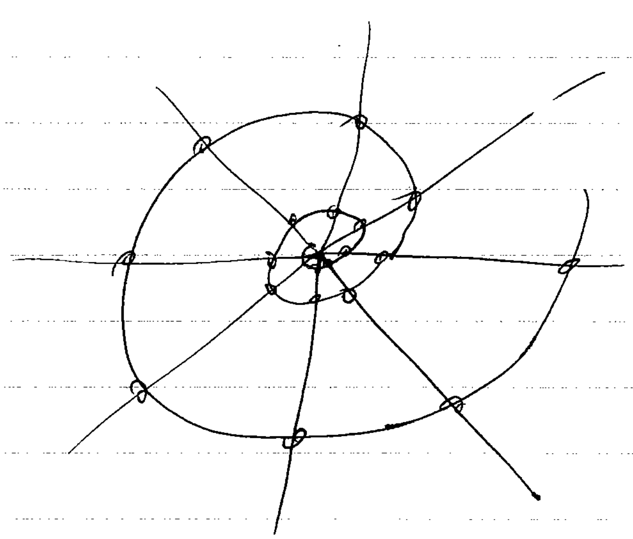


1(c) $\quad 1+i = |1+i|\, e^{i\pi/4} = \sqrt{1^2+1^2}\, e^{i\pi/4} = \sqrt{2}\, e^{i\pi/4}$



Get a spiral of points, ...

everywhere spiral hits the lines $\theta = \pi/4$, $\theta = \pi/2$, $\theta = 0$, $\theta = 3\pi/4$, $\theta = \pi$, $\theta = 5\pi/4$, $\theta = 6\pi/4$, $\theta = \frac{7\pi}{4}$.



(d) $h : \mathbb{C} - \{0\} \longrightarrow S^1$

$: z \longmapsto \dfrac{z}{|z|}$

takes $\left(\sqrt{2}\, e^{i\pi/4}\right)^n$ to $\dfrac{\sqrt{2}^n \, e^{in\pi/4}}{\left|\sqrt{2}^n \, e^{in\pi/4}\right|}$

$$= \frac{\cancel{\sqrt{2}^n} \, e^{in\pi/4}}{\cancel{\sqrt{2}^n}} = e^{in\pi/4}$$

Therefore $\boxed{h(G_2) = G_1}$

<u>Q2</u> eg

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\nparallel$$

$g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$

$h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$

whereas

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$\therefore \quad gh \neq hg \quad \forall g, h \in SL(2, \mathbb{Z})$

Q3 (a) $I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_{n\times n}$ satisfies $I_n^T I_n = I_n$

So $I_n \in O(n)$, $O(n)$ has identity element!

If $A, B \in O(n) \rightarrow A^T A = I = A A^T$
& $B^T B = I = B B^T$

$$\Rightarrow (AB)^T (AB) = (B^T A^T)(AB) = B^T (A^T A) B$$
$$= B^T I B$$
$$= B^T B = I$$

and $(AB)(AB)^T = A(B B^T) A^T = A I A^T = A A^T = I$

$\Rightarrow AB \in O(n)$, $O(n)$ closed under mult$^{\underline{n}}$.

If $A \in O(n) \Rightarrow A^T \in O(n)$ & $A^T = A^{-1}$

since $A^T A = A A^T = I$

inverses exist in $O(n)$.

Finally matrix mult$^{\underline{n}}$ is associative $\Rightarrow$ mult$^{\underline{n}}$ of $O(n)$ matrices is associative

3(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2) \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T$

$$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix}$$

$\Rightarrow \quad a^2+c^2 = 1 = a^2+b^2 \quad \Rightarrow \quad c^2 = b^2 \Rightarrow \boxed{c = \pm b.}$

and

$a^2+c^2 = 1 = c^2+d^2 \quad \Rightarrow \quad a^2 = d^2 \Rightarrow \boxed{d = \pm a}$

Also $ab+cd = 0$

$\Rightarrow \boxed{cd = -ab}$

$\Downarrow$

If $c = b$

then $d = -a$

If $c = -b$

then $d = +a$.

Two types of matrices $\Leftarrow$

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & -\cancel{b} \\ c & a \end{pmatrix}$$

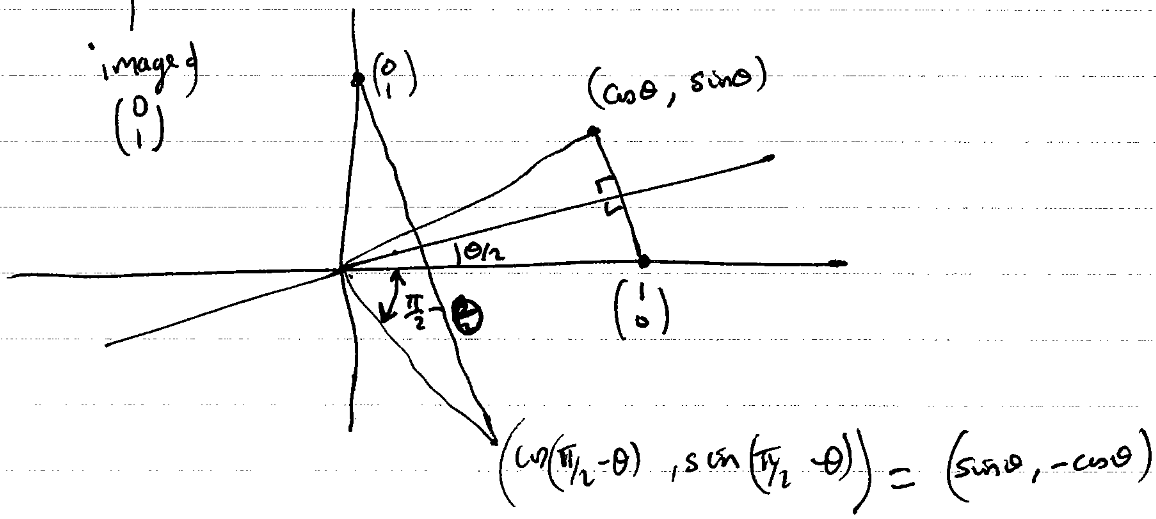
and $a^2+b^2 = a^2+c^2 = 1$.

3(c) Writing $a = \cos\theta \quad b = \sin\theta \Rightarrow$ get

$\boxed{\text{Linear Map}}$ $\hookrightarrow$ $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

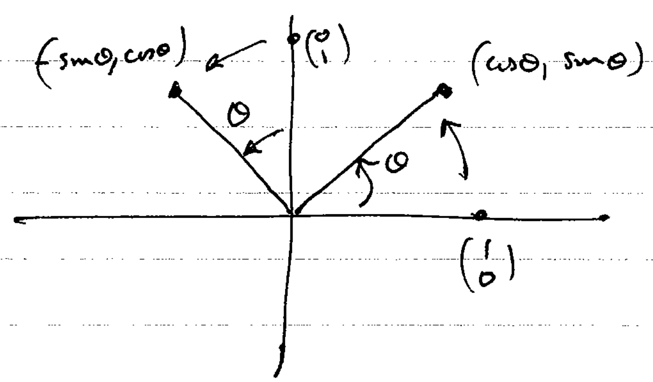
image of $\begin{pmatrix}1\\0\end{pmatrix}$ $\qquad$ image of $\begin{pmatrix}0\\1\end{pmatrix}$

• Reflection in the line through $(0,0)$ making angle $\frac{\theta}{2}$ with $\oplus$ x-axis



$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

image of $\begin{pmatrix}1\\0\end{pmatrix}$ $\qquad$ image of $\begin{pmatrix}0\\1\end{pmatrix}$



Rotation about $(0,0)$ through (counterclockwise) angle θ.

3(d) We know... $\det(AB) = \det(A)\det(B)$ --- from linear algebra

Also $\quad A \in O(2) \Rightarrow \quad \det\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = -(a^2+b^2) = -1$

$\quad$ or $\quad \det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = (a^2+b^2) = +1$

If A, B have $\det = 1$ then $\det(AB) = 1.1 = 1$

& $\det(A^T) = \det(A) = 1$

$$\underset{A^T = A^{-1}}{\uparrow}$$

$\therefore \quad SO(2) = \{ A \in O(2) \mid \det(A) = 1 \}$

is closed under mult^n & taking inverses

& clearly $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SO(2)$

$\Rightarrow \quad SO(2)$ is a subgroup of $O(2)$.
