

## Homework Set 3

Please complete by class time on Thursday, Feb 25.

1. Write down five different elements  $g \in S_5$  which conjugate  $(12)(34)$  into  $(13)(24)$ . That is, find five different elements  $g \in S_5$  which satisfy the equation

$$g(12)(34)g^{-1} = (13)(24)$$

2. Write down a detailed argument to show that the  $(n-1)n/2$  transpositions  $(pq)$  for  $1 \leq p < q \leq n$  generate all of  $S_n$ .
3. Write down a detailed argument to show that the two elements  $(12)$  and  $(1\dots n)$  generate all of  $S_n$ .
4. Verify that  $\{(12), (123)\}$ ,  $\{(12), (23)\}$  are two generating sets for  $S_3$ . Also, draw the Cayley graphs of  $S_3$  with respect to these two generating sets. You should draw two separate graphs.
5. Compare the Cayley graph of  $S_3$  with respect to  $\{(12), (123)\}$  with the Cayley graph of  $\mathbb{Z}_6$  with respect to  $\{2, 3\}$ . Any similarities? Any differences?
6. Draw the Cayley graph of  $S_4$  with respect to the generating set  $\{(12), (23), (34)\}$  (also verify that this is indeed a generating set).

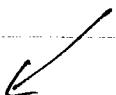
BASIC FACT ABOUT CONJUGATION. . .

1.  $g(12)(34)g^{-1} = (g(1)g(2))(g(3)g(4))$

Therefore  $g(12)(34)g^{-1} = (13)(24)$

becomes

$$(g(1)g(2))(g(3)g(4)) = (13)(24)$$



$$\begin{array}{ll} g(1)=1 & g(3)=2 \\ g(2)=3 & g(4)=4 \end{array}$$

$$\begin{array}{ll} g(1)=3 & g(3)=2 \\ g(2)=1 & g(4)=4 \end{array}$$

eg  $g = (23)(1)(4)(5)$

eg  $g = (132)(4)(5)$

$$\begin{array}{ll} g(1)=1 & g(3)=4 \\ g(2)=3 & g(4)=2 \end{array}$$

$$\begin{array}{ll} g(1)=3 & g(3)=4 \\ g(2)=1 & g(4)=2 \end{array}$$

eg  $g = (234)(1)(5)$

$$g = (2134)(5)$$

There are no other choices for  $g$ ! In all 4 cases, the <sup>effect of  $g$  on the</sup> remaining elements (including 5) ~~are~~ completely determined.

There are only 4 such  $g$ .

2. Effect of a permutation on  $\{1, \dots, n\}$  is to permute elements around in disjoint cycles. - - -

In cycle notation

$$\text{perm} = (m_1 \dots m_k)(p_1 \dots p_\ell) \dots$$

So it suffices to prove that each cycle can be written as a product of transpositions.

$$\text{But } (a_1 \dots a_n) = (a_1 a_2)(a_2 a_3) \dots (a_{n-1} a_n).$$

3. By Q2, it is sufficient to show that all  $n(n-1)/2$  transpositions can be obtained from  $(12)$  and  $g = (12 \dots n)$ .

$$\left. \begin{array}{l} g(12)g^{-1} = (23) \\ g^2(12)g^{-2} = (34) \\ \vdots \\ g^{n-2}(12)g^{-(n-2)} = (n-1, n) \\ g^{n-1}(12)g^{-(n-1)} = (n, 1) \end{array} \right\} \Rightarrow \text{have } (12), (23), \dots, (n-1, n), (n, 1)$$

$$(12)(23) = (123) = h_2$$

$$(12)(23)(34) = (1234) = h_3$$

$$\vdots$$

$$(12)(23) \dots (n-2, n-1) = (12 \dots n) = h_{n-2}$$

Note that  $h_1^{-1}(12) h_2 = (31) = (13)$   
 $h_2^{-1}(12) h_3 = (41) = (14)$   
 $\vdots$   
 $h_{n-1}^{-1}(12) h_{n-2} = (n-1) = (1 n-1)$

] - [ ] ↑  
 adding  
 $(12)$   
 $\& (1n)$   
 from earlier.

Now, to get  $(pq)$   $p < q$  we simply conjugate

$(1 (q-p+1))$  from list [ ] ~~+~~

by  $g^{p-1}$

$$g^{p-1} (1 (q-p+1)) g^{-(p-1)} = (pq)$$

Recall  $g = (12 \dots n)$

So now we have obtained all  $n(n-1)/2$  transpositions

$(pq)$   $p < q$  from  $(12)$  &  $(12 \dots n)$ .

Since the transpositions generate  $S_n \Rightarrow$  so does

$$\{(12), (12 \dots n)\}$$

4.  $\{(12), (123)\}$  works by Q3 ( $n=3$  case).

Given  $\{(12), (123)\} \Rightarrow$  can get  $(123) = (12)(23)$

$\Rightarrow \{(12), (123)\}$  generates same subgroup as  
 $\{(12), (123)\}$   
(i.e. all of  $S_3$ ).

Cayley graphs drawn in class!

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5. Done in class!

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6. Done in class!

Note:  $(12)(23)(34) = (1234) \Rightarrow$  we can get  
 $(12), (234)$

& hence all of  $S_4$

from  $(12), (23), (34)$ .

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