MATH 4513-001 Senior Seminar Spring 2010

Homework Set 2

Please complete by class time on Thursday, Feb 11.

1. Consider the multiplicative group $G_1 = \{e^{n\pi i/4} \,|\, n \in \mathbb{Z}\}$.
   (a) Is it finite or infinite?
   (b) Draw it on the unit circle $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$.
   (c) Plot some of the elements of the multiplicative group $G_2 = \{(1+i)^n \,|\, n \in \mathbb{Z}\}$ in the complex plane (minus the origin) At least, plot the values of n between -7 and 9. Hint: it's is easier if you first write $(1+i)$ in polar form.
   (d) What is the image of G_2 under the homomorphism

$$h : \mathbb{C} - \{0\} \to S^1 : z \mapsto \frac{z}{|z|}?$$

2. A group G is said to be *abelian* if the operation is *commutative*. That is $gh = hg$ for all $g, h \in G$. Show that the group $\mathrm{SL}(2, \mathbb{Z})$ is not abelian.

3. An $n \times n$ matrix A is said to be *orthogonal* if $A^T A = AA^T = I_n$, where I_n denotes the identity matrix (1's on the diagonal and 0's elsewhere) and A^T denotes the *transpose* of A (which is defined by requiring that its ij-entry is the ji-entry of A).
   (a) Check that the set $\mathrm{O}(n)$ of all $n \times n$ orthogonal matrices with real number entries is a group under matrix multiplication.
   (b) Prove that the elements of $\mathrm{O}(2)$ are either of the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

   where $a^2 + b^2 = 1$.
   (c) Writing $a = \cos\theta$ and $b = \sin\theta$ above, give a geometric interpretation of the two types of elements of $\mathrm{O}(2)$.
   (d) Verify that $\mathrm{SO}(2) = \{A \in \mathrm{O}(2) \,|\, |A| = 1\}$ is a subgroup of $\mathrm{O}(2)$. It is called the *special orthogonal group*.