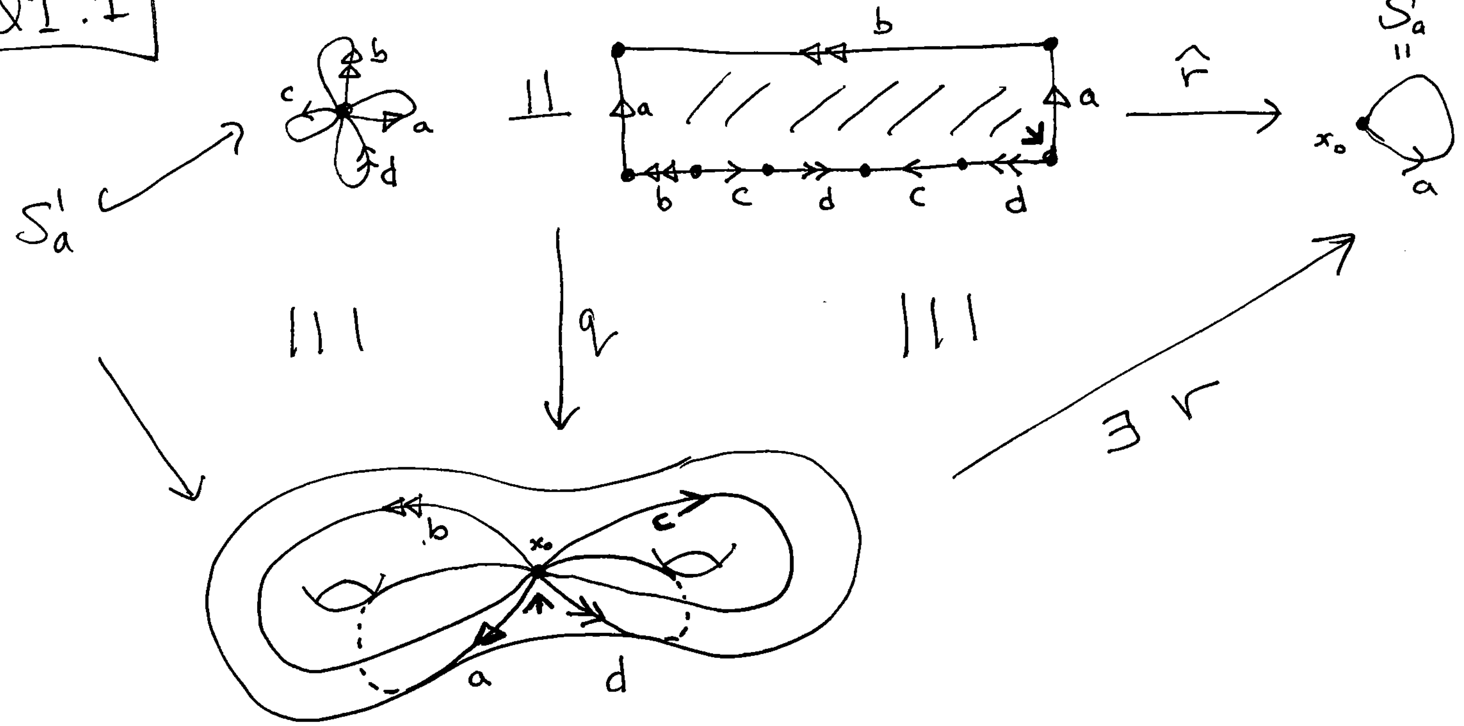
Q1.1

[METHOD FROM CLASS NOTES (ON HWK)]



Define $\hat{r} : S^1_a \vee S^1_b \vee S^1_c \vee S^1_d \amalg D^2 \longrightarrow S^1_a$ as follows:

$\hat{r}|_{S^1_a} = \mathbb{1}_{S^1_a}$

$$\left.\begin{array}{l} \hat{r}|_{S^1_b} \\ \hat{r}|_{S^1_c} \\ \hat{r}|_{S^1_d} \end{array}\right\} = C_{x_0}$$

constant map with image $\{x_0\} \subseteq S^1_a$.

and $\hat{r}|_{D^2} = q_1 \circ P_r$ where $P_r : D^2 \longrightarrow I_a =$  is horizontal (parallel to) projection onto the vertical segment I_a, and $q_1 : I_a \longrightarrow S^1_a$ is the standard quotient $[0,1] \to S^1$.

By construction, $\hat{r}$ is constant on the fibers of q_1 and so induces a well-defined cts map $r : \Sigma_2 \longrightarrow S^1_a$. Tracing through the composition with $S^1_a \hookrightarrow \Sigma_2$, we see that r is a retraction.

$\boxed{\text{Q1.1}}$ [ALTERNATIVE RETRACTION] ⒤

define $r : \Sigma_2 \to S^1_a$ by defining $r|_{\Sigma_2^{(1)}}$

$$r|_{\Sigma_2^{(1)}} : x_0 \longmapsto x_0$$

$$\left.\begin{aligned} : S^1_a &\longmapsto S^1_a \\ S^1_b &\longmapsto S^1_a \\ S^1_c &\longmapsto S^1_a \\ S^1_d &\longmapsto S^1_a \end{aligned}\right) \text{ homeomorphisms.} \quad \text{—} (*)$$

The path hom. class of the boundary of the 2-cell of Σ_2 is $[a,b][c,d]$ which gets sent to $\mathbb{Z} = \pi_1(S^1_a)$ as

$$r_*\left([a,b][c,d]\right)$$

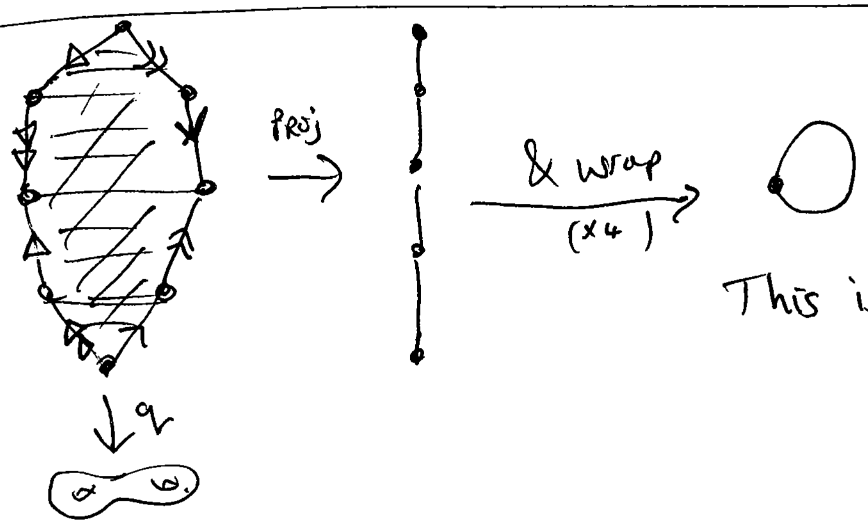
$$= [r_*(a), r_*(b)][r_*(c), r_*(d)]$$

$$= [a,a][a,a] \quad \text{- - - by } (*) \text{ above.}$$

$$= 1$$

$\Rightarrow$ r extends over the 2-cell! done.

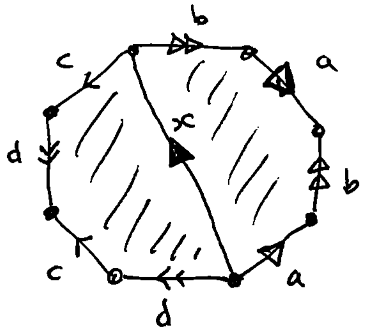
[ALTERNATIVE] (II)

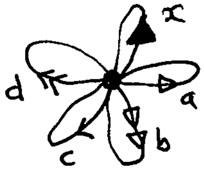


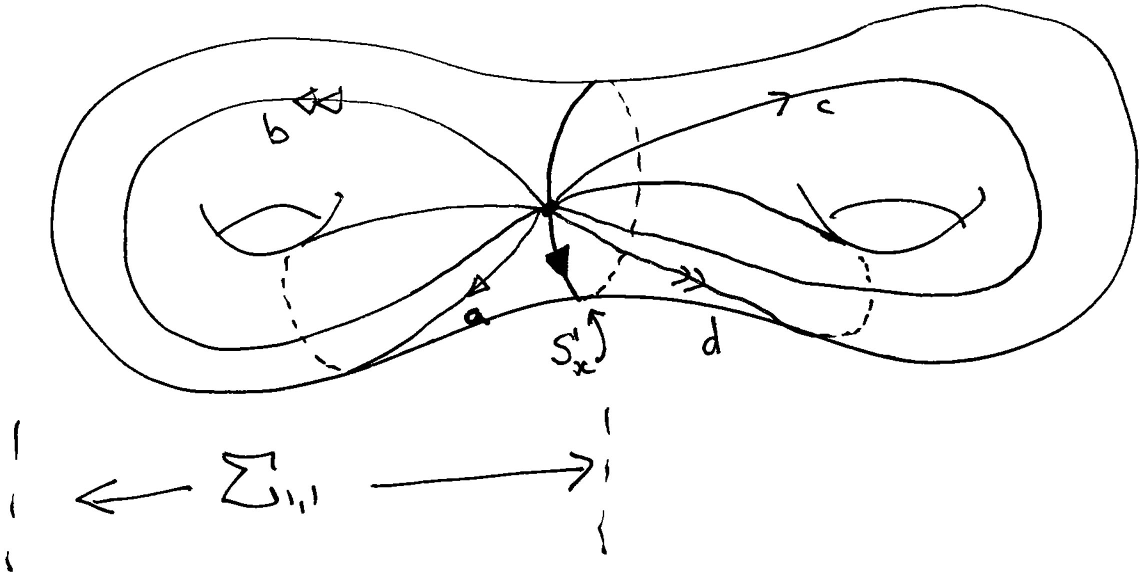
Composite is constant on q-fibers
$\Rightarrow \exists$ retraction
$r : \Sigma_2 \to S^1_a$
etc---

This is
$a \mapsto a$
$b \mapsto a$
$c \mapsto \bar{a}$
$d \mapsto \bar{a}$

Q1.2







Cutting Σ_2 along S'_x produces two subsurfaces, each of which is genus one with one hole. Denote the subsurface containing S'_a, S'_b, S'_x by $\Sigma'_{1,1}$.

$$\begin{tikzcd} S'_x \arrow[r, phantom, "\subseteq"] \arrow[rrr, bend left, "\mathbb{1}_{S'_x}"] & \Sigma_{1,1} \arrow[r, phantom, "\subseteq"] & \Sigma_2 \arrow[r, dashed, "?\exists r"'] & S'_x \end{tikzcd}$$

(*) — If $\exists$ retract $r: \Sigma_2 \to S'_x$, then composing with the inclusion $\Sigma_{11} \subseteq \Sigma_2$ gives a retract $r = r|_{\Sigma_{1,1}} : \Sigma_{1,1} \to S'_x$

We'll show $\nexists$ retraction $\Sigma_{1,1} \longrightarrow S^1_x$

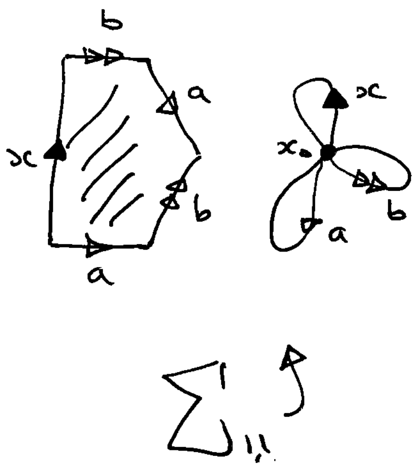
(& hence $\nexists$ retraction $\Sigma_2 \to S^1_x$ by (✱) above).

<u>Fact 1</u> $\pi_1(\Sigma_{1,1}, x_0) \cong F_{\{a,b\}}$ and $x = aba^{-1}b^{-1} = [a,b]$.

<u>Pf</u> Use S.v.K. theorem.

$$\pi_1(\Sigma_{1,1}, x_0) = \langle a, b, x \mid xbab^{-1}a^{-1} \rangle$$
$$= \langle a, b, x \mid x = aba^{-1}b^{-1} \rangle$$
$$= \langle a, b \mid \rangle$$
$$\cong F_{\{a,b\}} \quad \text{and} \quad x = [a,b]$$

done! ▨



Now go for the usual contradiction ....

If $\exists$ retraction, then get group retraction

$$\begin{tikzcd} & (\Sigma_{1,1}, x_0) \arrow[dr, "r"] & \\ (S^1_x, x_0) \arrow[ur, "i"] \arrow[rr, "\mathbb{1}_{S^1_x}"'] \arrow[rr, phantom, shift left=4ex, "///"] & & (S^1_x, x_0) \end{tikzcd} \Longrightarrow \begin{tikzcd} & \pi_1(\Sigma_{1,1}, x_0) \arrow[dr, "r_*"] & \\ \pi_1(S^1_x, x_0) \arrow[ur, "i_*"] \arrow[rr, "\mathbb{1}_{\pi_1(S^1_x, x_0)}"'] \arrow[rr, phantom, shift left=4ex, "///"] & & \pi_1(S^1_x, x_0) \end{tikzcd}$$

In other words we have a retraction

$$\begin{tikzcd} & F_{\{a,b\}} \arrow[dr, "r_*"] & \\ \mathbb{Z} \cong \langle x \mid \rangle \arrow[ur, "i_*"] \arrow[rr, "\mathbb{1}_{\mathbb{Z}}"'] \arrow[rr, phantom, shift left=4ex, "///"] & & \langle x \mid \rangle \cong \mathbb{Z} \end{tikzcd}$$

but $i_*(x) = [a,b]$

& so $r_* \circ i_*(x) = [r_*(a), r_*(b)]$

$= 1$ in $\underline{\mathbb{Z}x} | \rangle \cong \mathbb{Z}$ abelian

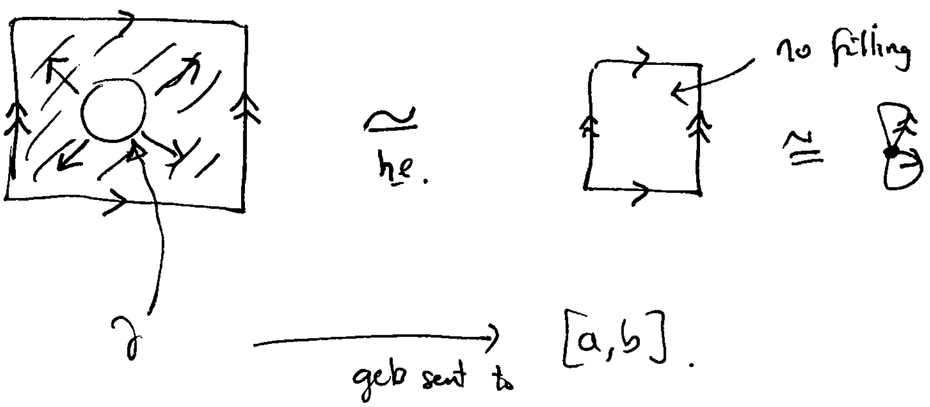
$\boxed{x \neq 1 \text{ in } \langle x | \rangle = \mathbb{Z}}$

$\neq \mathbb{1}_{\mathbb{Z}}(x)$

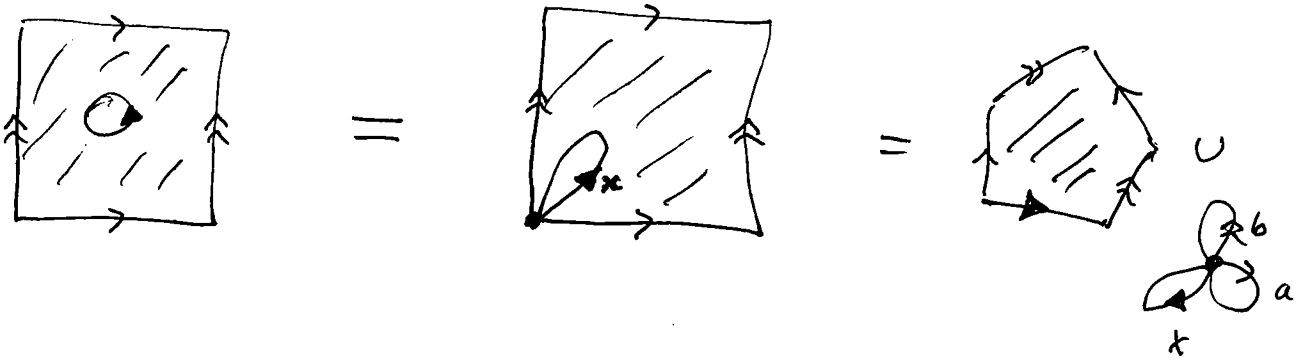
$\Rightarrow$ contradiction!

Remark: This "three 1-cells & one pentagonal 2-cell" gives a much more efficient/clean approach to the result that $\nexists$ retraction of punctured torus onto its boundary circle.

Before we worked with retractions onto $S^1_a \vee S^1_b$ & worried (ever so slightly!) about base points etc..



Our New perspective is to "push the hole to one corner" and work with one new generator and a pentagonal relator ...



Comment on Q 2.2 (a)

People had problems with product cell structure.

It's just a cartesian product !

$I = [0,1]$ 0-cells 0 & 1

1-cell I .

I^3 has 8 0-cells $(0,0,0), (0,0,1), \text{----}, (1,1,1)$

12 1-cells $0\times 0\times I$, $0\times 1\times I$, $1\times 0\times I$, $1\times 1\times I$,

$0\times I\times 0$, $0\times I\times 1$, $1\times I\times 0$, $1\times I\times 1$,

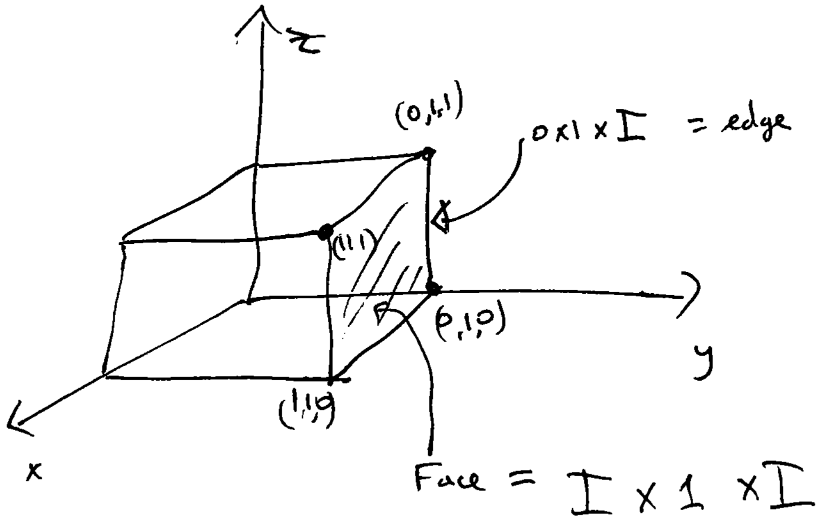
$I\times 0\times 0$, $I\times 0\times 1$, $I\times 1\times 0$, $I\times 1\times 1$

6 2-cells $0\times I\times I$, $1\times I\times I$,

$I\times 0\times I$, $I\times 1\times I$,

$I\times I\times 0$, $I\times I\times 1$

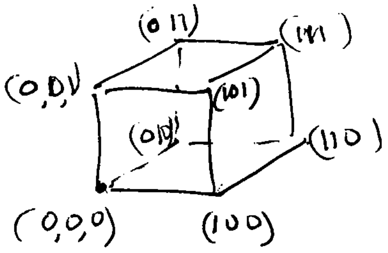
& 1 3-cell I^3 & 0 m-cells for $m \geq 4$.



ex !

Fill in the rest !

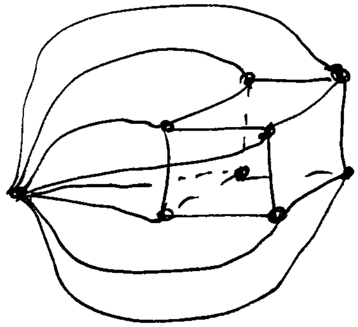
$\boxed{Q2.2}$ (a)



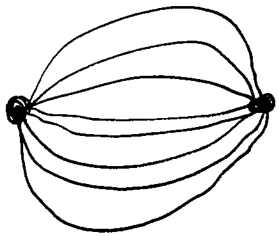
8	0-cells
12	1-cells
6	2-cells
1	3-cells
0	m-cells $(m \geq 4)$

$\boxed{Q2.2}$ (b) [Alternative method]

$Z = C(8) \cup_8 I^3 =$



$Z/_{I^3} =$



$\simeq \bigvee_{i=1}^{7} S^1$

S/he ↶

Z (since I^3 is a contractible subcomplex of Z)

$\int$ | he ↶ (since $C(8)$ is a contractible subcomplex of Z)

$Z/_{C(8)}$

But $Z/_{C(8)} = I^3/$vertices identified

$\pi_1(Z) = \pi_1(Z/_{I^3})$

$= \pi_1\left(\bigvee_{i=1}^{7} S^1\right)$

$= F_7$ free group of rank 7.

π_1(cube w/ vertices identified)

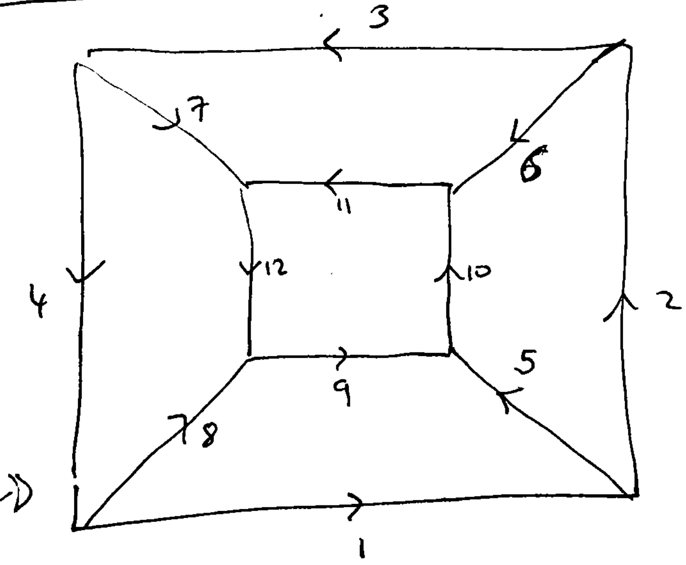
$= \pi_1(Z/_{C(8)})$

$= \pi_1(Z)$

$= \pi_1(Z/_{I^3})$

$= F_7$ from above.

$\boxed{Q\,2.2}$ (b) [STRAIGHT ON APPROACH]



identify all 0-cells to get X

$\Rightarrow X^{(1)}$ is a bouquet of 12 circles

& $X^{(2)}$ has 6 2-cells.

$$\pi_1(X) = \pi_1(X^{(2)}) = \langle a_1, \text{—}, a_{12} \mid a_1a_2a_3a_4,\ a_9a_{10}a_{11}a_{12},\ a_2a_6 = a_5a_{10},\ a_3a_7 = a_6a_{11},\ a_4a_8 = a_7a_{12},\ a_1a_5 = a_8a_9 \rangle$$

It is clear from the diagram above that $a_1a_2a_3a_4$ is <u>a consequence</u> of the other 5 relators $\Rightarrow$ we can omit it. (Teitze II)

$$\Rightarrow \pi_1(X^{(2)}) = \langle a_1 \text{—} a_{12} \mid a_9a_{10}a_{11}a_{12},\ a_1 = a_8a_9\bar{a}_5,\ a_2 = a_5a_{10}\bar{a}_6,\ a_3 = a_6a_{11}\bar{a}_7,\ a_4 = a_7a_{12}\bar{a}_8 \rangle$$

These enable us to remove a_1, a_2, a_3 & a_4 ...

(Teitze I)

$$= \langle a_5, \text{—}, a_{12} \mid (a_9a_{10}a_{11})^{-1} = a_{12} \rangle$$

This enables us to remove a_{12}.

$$= \langle a_5, \text{—}, a_{11} \mid \ \rangle$$

$$\cong F_7$$

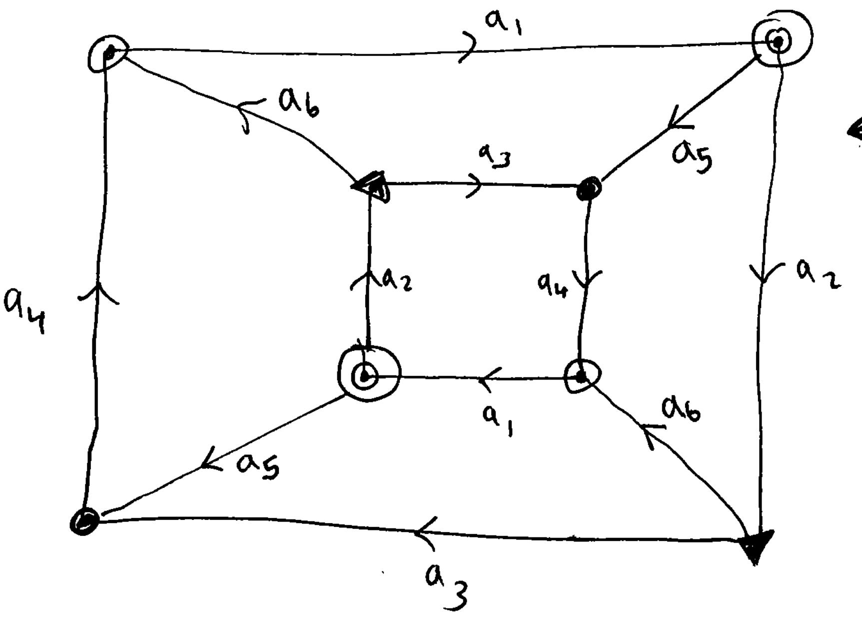
$\boxed{Q\ 2.3}$ [SLICK WAY] Nobody said this ☹

This space is just cube/antipodal map on ∂(cube) $\cong \mathbb{R}P^3$, has $\pi_1 = \mathbb{Z}_2$.

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 : \vec{x} \longmapsto -\vec{x}$ & "cube" $= [-1,1]^3$

"antipodal map" on boundary of cube $\rightarrow f|_{\partial([-1,1]^3)}$

[(SLOW) DIRECT WAY] OVERHEAD VIEW



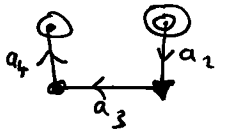
↔

Key idea! ① First identify "Top" $a_1a_2a_3a_4$ square with "BOTTOM" $a_1a_2a_3a_4$, as shown.

② Now note that side face pairings dont ~~effect~~ make any more identifications of the 4 0-cells or the 4 1-cells a_1, a_2, a_3, a_4. ... just some pairings of the remaining 4 1-cells connecting TOP to BOTTOM.

Quotient space, Y, has 4 0-cells, 6 1-cells, 3 2-cells & one 3-cell.

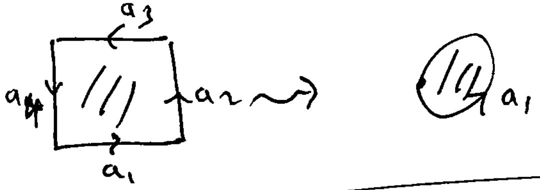
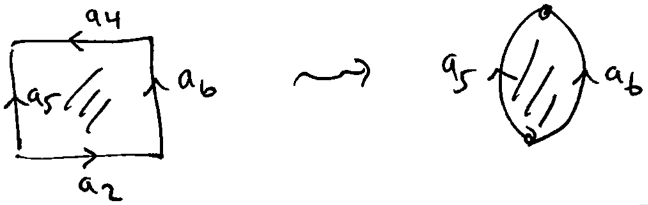
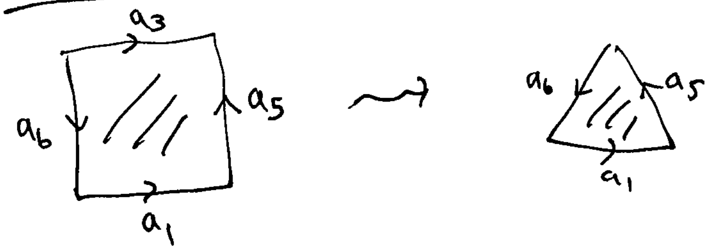
$\pi_1(Y) = \pi_1(Y^{(2)})$ ← Collapse maximal tree, eg $T =$



, in $Y^{(1)}$ and apply S.v.K.

to get

$$\pi_1(Y) = \langle a_1, a_5, a_6 \mid a_1,\ a_5 = a_6,\ a_1 a_5 a_6 \rangle \longrightarrow \text{why?}$$

Effect of collapsing T	Resulting Relation
	a_1
	$a_5 = a_6$
	$a_1 a_5 a_6$

$$\pi_1(Y) = \langle a_5, a_6 \mid a_5 = a_6 , \ a_5 a_6 = 1 \rangle$$

$$= \langle a_5 \mid a_5^2 = 1 \rangle \cong \mathbb{Z}_2$$

Q3.1

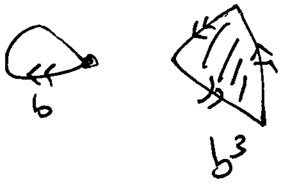
$$\pi_1(B) = \langle a \mid a^4 \rangle \cong \mathbb{Z}_4$$

$$\begin{aligned}\pi_1(Y) &= \langle a \mid a^4, a^2 \rangle \\ &= \langle a \mid a^2, (a^2)^2 \rangle \\ &\cong \langle a \mid a^2 \rangle \cong \mathbb{Z}_2\end{aligned}$$

$$\begin{aligned} i_* : \pi_1(B) &\longrightarrow \pi_1(Y) \\ : \mathbb{Z}_4 &\longrightarrow \mathbb{Z}_2 \\ \langle a \mid a^4 \rangle &\longrightarrow \langle a \mid a^2 \rangle \\ a &\longmapsto a \end{aligned}$$

gen. ↦ gen.

Q3.2 Let $X =$

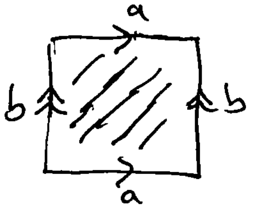


so $\pi_1(X) = \mathbb{Z}_3$

Then $Y \vee X$ has $\pi_1(Y \vee X) \cong \mathbb{Z}_2 * \mathbb{Z}_3$

and $Y \vee X$ has a subcomplex $B \vee X$ which has $\pi(B \vee X) \cong \mathbb{Z}_4 * \mathbb{Z}_3$

Let $Z = Y \vee X$ with the 2-cell



attached.

Then S-vK thm $\Rightarrow$ $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_3$.

The inclusion map

$$B \vee X \xhookrightarrow{i} Z$$

induces the homomorphism

$$i_* : \mathbb{Z}_4 * \mathbb{Z}_3 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_3$$

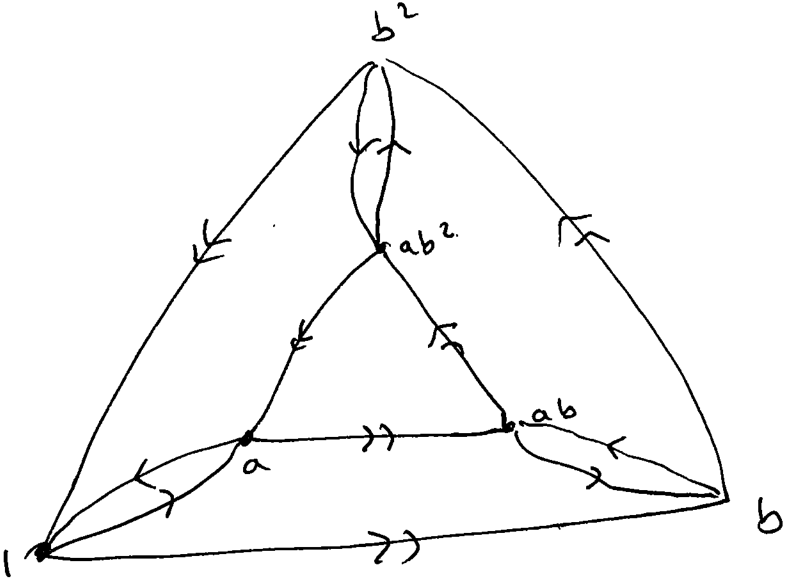
which takes a gen of $\mathbb{Z}_4$ to a gen of $\mathbb{Z}_2$
& takes a gen of $\mathbb{Z}_3$ to a gen of $\mathbb{Z}_3$.

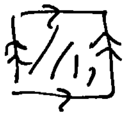
Let $\tilde{Z}$ denote the (simply connected) universal cover of Z. Q III.2 $\Rightarrow$

$$\begin{tikzcd}
p^{-1}(B \vee X) = \widehat{B \vee X} \arrow[r, hook, "j"] \arrow[d, "p"] & \tilde{Z} \arrow[d, "p"] \\
B \vee X \arrow[r, hook, "i"] & Z
\end{tikzcd}$$

$$p_*\left(\pi_1\left(\widehat{B \vee X}\right)\right) = \mathrm{Ker}(i_*)$$

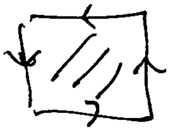
But $\tilde{Z}$ is a standard 6-fold covering space of Z (since $\pi_1(Z) = \mathbb{Z}_2 \times \mathbb{Z}_3$ has order 6), whose 1-skeleton we've drawn many times!



We obtain $\tilde{Z}$ by attaching 6 copies of , 6 copies of , 6 copies of , and 6 copies of  to this 1-skeleton.

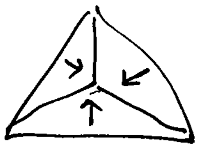
We obtain $\widehat{B \vee X}$ by attaching 6 copies of  and 6 copies of  to this 1-skeleton.

✓

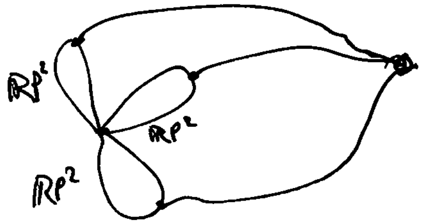
For π_1 computation purposes we can just attach 2 copies of  (one per cycle of  b-edges) and 3 copies of  (one per cycle ~~of~~ of a edges).

This gives 3 copies of $\mathbb{RP}^2$ connected to 2 triangles.

Collapse one triangle ~~to get~~ to a vertex, and collapse the other triangle to a tripod



to get a h.e. space



This is h.e. to

$\left(\bigvee_{i=1}^{3} \mathbb{RP}^2\right) \vee$ 

$\underset{h.e.}{\simeq} \left(\bigvee_{i=1}^{3} \mathbb{RP}^2\right) \vee$ 

$$\overset{\ker(i_*)}{\overset{\shortparallel}{\pi}} \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * F_2$$

Which is in keeping with the Kurosh theorem.

Abstract description

We get a concrete description by picking appropriate edge path loops from $(*)$ eg...

$$\mathrm{Ker}(i_*) = \langle a^2\rangle * b\langle a^2\rangle b^{-1} * b^2\langle a^2\rangle b^{-2} * \langle \quad \rangle$$

$[a,b]$, $[a,\bar{b}]$

This is an explicit set of generators, as words in $\mathbb{Z}_4 * \mathbb{Z}_3 = \langle a \mid a^4\rangle * \langle b \mid b^3\rangle$

Now we have <u>explicit info</u> for Kurosh:

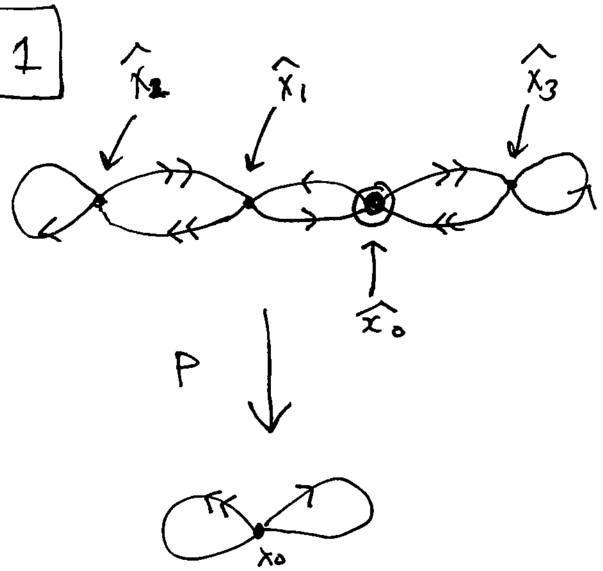
we have 3 distinct conjugates of the $\mathbb{Z}_2$-subgroup $\langle a^2\rangle$ of $\mathbb{Z}_4$ in $\mathbb{Z}_4 * \mathbb{Z}_3$

$(1\langle a^2\rangle 1^{-1},\ b\langle a^2\rangle b^{-1},\ b^2\langle a^2\rangle b^{-2})$

And an F_2 generated by the commutators

$[a,b]$ and $[a,b^2]$

Q.4.1



After some easy folding we find that H corresponds to the following covering space

Q4.2

$$H = \pi_1(\hat{X}) \simeq \pi_1\left(\bigvee_{i=1}^{5} S^1\right) \cong \underset{\sim}{F}_5$$

graph → graph/max'l tree

(The given generators are in fact a free basis for H.)

Q4.3

$$[F_{\{a,b\}} : H] = \text{degree (\#sheets) of covering space } \hat{X} \xrightarrow{p} X$$

$$= 4$$

Q4.4

lift the loop $a^3 b^3 a^{-3}$ to a path in $\hat{X}$ based at $\hat{x}_0$

$\Rightarrow a^3$ ends at $\hat{x}_1$

$\Rightarrow a^3 b^3$ ends at $\hat{x}_2$

$\Rightarrow a^3 b^3 a^{-3}$ ends at $\hat{x}_2$ too.

$\Rightarrow$ Not a loop

$\Rightarrow a^3 b^3 a^{-3} \notin H$.

$\boxed{\text{Q4.5}}$ We know from class notes that

$$N_{F_2}(H)/H \cong \mathrm{Aut}(\hat{X} \xrightarrow{p} X).$$

So, determining $N_{F_2}(H)$ is equivalent to determining $\mathrm{Aut}(\hat{X} \xrightarrow{p} X)$.

∄ deck transformation of $\hat{X}$ taking $\hat{x}_0$ to $\hat{x}_3$.

Reason: ∄ "a-loop" based here, **but** ∃ "a-loop" based here.

Likewise, ∄ deck transformation of $\hat{X}$ taking $\hat{x}_0$ to $\hat{x}_2$.

However, ∃! deck transformation taking $\hat{x}_0$ to $\hat{x}_0$ (namely $\mathbb{1}_{\hat{X}}$)

and ∃! - - - - - - taking $\hat{x}_0$ to $\hat{x}_1$
(namely, $\hat{x}_2 \leftrightarrow \hat{x}_3$, $\hat{x}_0 \leftrightarrow \hat{x}_1$)

$$\Rightarrow \quad p_*(\pi_1(\hat{X}, \hat{x}_1)) = H = p_*(\pi_1(\hat{X}, \hat{x}_0))$$

but $\hat{x}_1 = \hat{x}_0 \cdot a$

$$\Rightarrow \quad p_*(\pi_1(\hat{X}, \hat{x}_1)) = a^{-1}\, p_*(\pi_1(\hat{X}, \hat{x}_0))\, a$$

$$= a^{-1} H a$$

$$\Rightarrow \quad a^{-1} H a = H \quad \Rightarrow \quad a \in N_{F_2}(H).$$

Since $\mathrm{Aut}(\hat{X} \xrightarrow{p} X) = \mathbb{Z}_2$ (only has 2 elements!)

we have that

$$\boxed{N_{F_2}(H)/H \cong \mathbb{Z}_2}$$

& so $N_{F_2}(H)$ contains H as an index 2 subgp

$$\Rightarrow \boxed{\begin{aligned} N_{F_2}(H) &= \langle a, H \rangle \\ &= \langle a, a^2, b^2, bab, ab^2a, ababa \rangle \\ &= \langle a, b^2, bab \rangle \end{aligned}}$$

$N_{F_2}(H)$ cleared up a bit!

Comments on [Q 4.5] . . .

Some of you said

$$N_G(H) = \langle\langle H \rangle\rangle$$

or that, knowing $a \in N_G(H)$, - - -

$$N_G(H) = \langle\langle a, H \rangle\rangle.$$

This is <u>NOT</u> true in general.

A normalizer of a subgroup H of a group G does not have to be a normal subgroup of G.

It is safest to say $N_{F_2}(H) = \langle a, H \rangle$

we know this because:

① $N_{F_2}(H)/_H \cong \mathbb{Z}_2$

& so $N_{F_2}(H)$ is an index 2 <u>super group</u> of H.

② & $a \in N_{F_2}(H) - H$.

IN this special example, $[F_2 : N_{F_2}(H)] = 2$ & so $N_{F_2}(H) \lhd F_2$ but this is coincidence!!