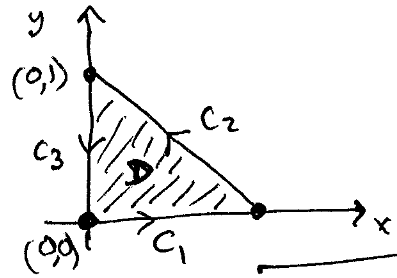
Q1]... [25 points] Compute the line integral

$$\oint_C y\,dx - 2x\,dy$$

where C is the closed, piecewise straight line path connecting $\underbrace{(0,0) \text{ to } (1,0)}_{C_1}$, then $\underbrace{(1,0) \text{ to } (0,1)}_{C_2}$, and then $\underbrace{(0,1) \text{ back to } (0,0)}_{C_3}$. Show your work.

Step ① : Draw the curve C & enclosed region D.



$$\partial D = C = C_1 \cup C_2 \cup C_3$$

Step ② : Evaluate $\oint_C$ Method I (Direct)

$$\boxed{\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}}$$

Along C_1: $y = 0 \Rightarrow y\,dx = 0$; $\Downarrow$ $dy = 0 \Rightarrow -2x\,dy = 0$ $\Rightarrow \boxed{\int_{C_1} = 0}$

$$\boxed{\int_{C_2} = -\frac{3}{2}}$$

Along C_3: $x = 0 \Rightarrow -2x\,dy = 0$; $\Downarrow$ $dx = 0$ $\Rightarrow \boxed{\int_{C_3} = 0}$

$$\oint_C = 0 + \left(-\frac{3}{2}\right) + 0 = \boxed{-3/2}$$

Finally C_2: $\vec{r}(t) = (1-t)\langle 1,0\rangle + t\langle 0,1\rangle = \langle 1-t, t\rangle, \quad 0 \le t \le 1$

$x = 1-t \Rightarrow \frac{dx}{dt} = -1 \qquad y = t \Rightarrow \frac{dy}{dt} = 1$

$$\int_{C_2} y\,dx - 2x\,dy = \int_0^1 [t(-1) - 2(1-t)(1)]\,dt = \int_0^1 t - 2\,dt = \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{3}{2}$$

Step ② : Evaluate $\oint_C$ Method II (Using Green's Thm)

$$\oint_C (y)\,dx + (-2x)\,dy = \iint \frac{\partial}{\partial x}(-2x) - \frac{\partial}{\partial y}(y)\,dA = \iint_D (-2-1)\,dA$$

$$= -3\iint_D dA = -3\,\text{Area}(D) = -3\left(\tfrac{1}{2}\right) = \boxed{-\frac{3}{2}}$$

Δ with base 1 & height 1

GET SAME ANSWER (YAY!)

Q2]... [25 points] Throughout this question, let $\mathbf{F}$ be the vector field given by

$$\mathbf{F} = \left\langle \frac{2x}{z}, \frac{2y}{z}, -\frac{(x^2+y^2)}{z^2} \right\rangle$$

(a) Compute curl($\mathbf{F}$). Show your work.

$$\text{Curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2x}{z} & \frac{2y}{z} & -\frac{(x^2+y^2)}{z^2} \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\left(\frac{-(x^2+y^2)}{z^2}\right) - \frac{\partial}{\partial z}\left(\frac{2y}{z}\right), \frac{\partial}{\partial z}\left(\frac{2x}{z}\right) - \frac{\partial}{\partial x}\left(\frac{-(x^2+y^2)}{z^2}\right), \frac{\partial}{\partial x}\left(\frac{2y}{z}\right) - \frac{\partial}{\partial y}\left(\frac{2x}{z}\right) \right\rangle$$

$$= \left\langle \frac{-2y}{z^2} - \left(-\frac{2y}{z^2}\right), \frac{-2x}{z^2} - \left(-\frac{2x}{z^2}\right), 0 - 0 \right\rangle = \langle 0, 0, 0 \rangle$$

(b) Is there a function f such that $\nabla f = \mathbf{F}$? If not, say why not. If so, find one such function f. Show your work.

domain of $\vec{F}$ = all $\mathbb{R}^3$ minus xy-plane = union of 2 regions with "no holes".

So $\vec{F}$ is a gradient. $\leftarrow$ $\nabla f = \vec{F}$

$$\Rightarrow \left\{ \begin{array}{l} f_x = \frac{2x}{z} \\ f_y = \frac{2y}{z} \\ f_z = -\frac{(x^2+y^2)}{z^2} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} f = \frac{x^2}{z} + g_1(y,z) \\ f = \frac{y^2}{z} + g_2(x,z) \\ f = \frac{x^2+y^2}{z} + g_3(x,y) \end{array} \right\} \Rightarrow \boxed{f = \frac{x^2+y^2}{z} + \text{Const}}$$

(c) Find the work done by the field $\mathbf{F}$ above as it moves a particle along a straight line path from $(1,1,1)$ to $(3,4,5)$.

C

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \underset{\text{Fund. Th}^{m}}{=} f(3,4,5) - f(1,1,1)$$

$$= \frac{9+16}{5} - \frac{1+1}{1} = 5 - 2 = \boxed{3}$$

Q3]... [25 points] Show that the identity

$$\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0$$

holds for **all** vector fields $\mathbf{F} = \langle P, Q, R \rangle$.

$$\mathrm{div}\left(\mathrm{curl}(\vec{F})\right) = \mathrm{div}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \mathrm{div}\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$

$= 0$ by Clairaut (mixed p.d.s are equal)

Need P, Q, R have cts 2nd p.d.'s.

Show that the identity

$$\nabla \times (f\mathbf{F}) = (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F})$$

holds for **all** vector fields $\mathbf{F} = \langle P, Q, R \rangle$ and functions $f(x, y, z)$.

$$\nabla \times (f\vec{F}) = \nabla \times \langle fP, fQ, fR \rangle$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fP & fQ & fR \end{vmatrix}$$

$$= \langle (fR)_y - (fQ)_z, (fP)_z - (fR)_x, (fQ)_x - (fP)_y \rangle$$

Product Rule for $\frac{\partial}{\partial x}$ etc & gather ("like" terms...)

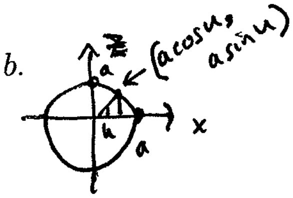
$$= f\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$
$$+ \langle f_y R - f_z Q, f_z P - f_x R, f_x Q - f_y P \rangle$$

$$= f(\nabla \times \vec{F}) + (\nabla f) \times \vec{F}$$

Q4]... [25 points] Write down a parametric description of the form

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$$

for the portion of the cylinder $x^2+z^2=a^2$ that lies between the planes $y=0$ and $y=b$.



Choose parameters --- $\begin{cases} u = \text{Polar angle in the } xz\text{-plane} \\ v = y\text{-coordinate} \end{cases}$ $\longleftrightarrow$

$$\vec{r}(u,v) = \langle a\cos u,\ v,\ a\sin u\rangle \qquad \begin{matrix} 0 \le u \le 2\pi \\ 0 \le v \le b. \end{matrix}$$

Using the area element for parametric surfaces, compute the area of the portion of the cylinder described above. Show your work.

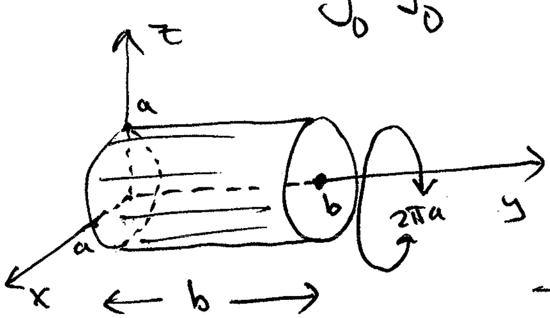
$$\vec{r}_u = \langle -a\sin u,\ 0,\ a\cos u\rangle$$

$$\vec{r}_v = \langle 0,\ 1,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle -a\cos u,\ 0,\ -a\sin u\rangle$$

$$ds = |\vec{r}_u \times \vec{r}_v|\,du\,dv = \sqrt{a^2(\cos^2 u+\sin^2 u)}\,du\,dv$$
$$= a\,du\,dv$$

$$\text{Area} = \int_0^b\int_0^{2\pi} a\,du\,dv = a\,[u]_0^{2\pi}\,[v]_0^b$$



$$= a\,(2\pi-0)(b-0)$$
$$= (2\pi a)\,b$$

Makes sense!